

From Weak to Strong Coupling in ABJM Theory

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Abstract: The partition function of $\mathcal{N} = 6$ supersymmetric Chern–Simons–matter theory (known as ABJM theory) on \mathbb{S}^3 , as well as certain Wilson loop observables, are captured by a zero dimensional super-matrix model. This super-matrix model is closely related to a matrix model describing topological Chern–Simons theory on a lens space. We explore further these recent observations and extract more exact results in ABJM theory from the matrix model. In particular we calculate the planar free energy, which matches at strong coupling the classical IIA supergravity action on $\text{AdS}_4 \times \mathbb{CP}^3$ and gives the correct $N^{3/2}$ scaling for the number of degrees of freedom of the M2 brane theory. Furthermore we find contributions coming from world-sheet instanton corrections in \mathbb{CP}^3 . We also calculate non-planar corrections, both to the free energy and to the Wilson loop expectation values. This matrix model appears also in the study of topological strings on a toric Calabi–Yau manifold, and an intriguing connection arises between the space of couplings of the planar ABJM theory and the moduli space of this Calabi–Yau. In particular it suggests that, in addition to the usual perturbative and strong coupling (AdS) expansions, a third natural expansion locus is the line where one of the two ’t Hooft couplings vanishes and the other is finite. This is the conifold locus of the Calabi–Yau, and leads to an expansion around topological Chern–Simons theory. We present some explicit results for the partition function and Wilson loop observables around this locus.

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1. Introduction and Summary

The discovery of Aharony, Bergman, Jafferis and Maldacena (ABJM) of the world-volume theory of coincident M2-branes [1] (following Bagger-Lambert and Gustavsson [2,3]) provides a new interacting field theory with well defined weak and strong coupling expansions. A great deal of effort has been given to studying these two limits of the theory: three dimensional $\mathcal{N} = 6$ supersymmetric Chern–Simons-matter and type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$ (or M-theory on $\text{AdS}_4 \times \mathbb{S}^7/\mathbb{Z}_k$). For better or worse, both descriptions of the theory are much harder than the D3-brane analog: 4d $\mathcal{N} = 4$ SYM and type IIB string theory on $\text{AdS}_5 \times \mathbb{S}^5$. At weak coupling perturbative calculations in ABJM theory are rather subtle and for many quantities are in even powers of the coupling, while at strong coupling the geometry of \mathbb{CP}^3 is more complicated than \mathbb{S}^5 and has, for example, non-trivial 2-cycles.

An important breakthrough, which is the underpinning of the present study, was the work of Kapustin, Willett and Yaakov [4], who use the localization techniques of [5] to reduce the calculation of certain quantities in the gauge theory on \mathbb{S}^3 to finite dimensional matrix integrals.¹ These matrix integrals can be evaluated in a systematic expansion in $1/N$. Indeed, they have a natural supergroup structure, *i.e.*, they are super-matrix models [7,8], and are related to some previously studied bosonic matrix models [9,10] by analytical continuation [8].

The solution of this matrix model allowed for the evaluation of the first exact interpolating function in this theory [8] giving a closed form expression for the expectation value of the $1/2$ BPS Wilson loop operator of [7] at all values of the coupling. This expression derived from the matrix-model reduction of the gauge theory reproduces exactly the known leading strong coupling result, the classical action of a macroscopic string in AdS_4 .

¹ Similar results apply also to other 3d theories with $\mathcal{N} = 2$ supersymmetry [4,6].

The purpose of this paper is to explore further what can be learnt from the matrix model and its solution to the understanding of the physical 3d gauge theory and its string/M-theory dual.

This is a broad subject, connected through the matrix model to special geometry, Chern–Simons (CS) theory, topological strings and more. One of the avenues we explore is the relation between the moduli space of the matrix model and the space of couplings of the gauge theory. It is very useful to consider the generalization of the gauge theory where the rank of the two gauge groups are not equal [11].² The space of couplings is two dimensional and upon complexification, it matches the moduli space of the Riemann surface solving the planar matrix model. This surface is also the mirror to a well studied toric Calabi–Yau manifold known as local \mathbb{F}_0 , where $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ is a Hirzebruch surface. As we review in Sect. 3, this moduli space has three special loci: the orbifold point, the large radius limit and the conifold locus.

These can be identified in the gauge theory respectively as the weakly coupled gauge theory, the strongly coupled theory described by string theory on AdS, and lastly the conifold locus is where the rank of one of the gauge group vanishes, so ABJM theory reduces to topological CS theory [12]. The first two are known duality frames with the AdS/CFT rules on how to evaluate observables on both sides. The simplicity of the conifold locus suggests that there should be another duality frame where ABJM theory is considered as a deformation of topological CS theory. We explore this in Sect. 6, where we calculate the partition function and Wilson loop observables around this point. It would be very interesting to learn how to calculate other quantities in this regime.

We present the matrix model for the ABJM theory and that for CS theory on the lens space $L(2, 1) = \mathbb{S}^3/\mathbb{Z}_2$ in the next section. The matrix model of ABJM has an underlying $U(N_1|N_2)$ symmetry while that of the lens space has $U(N_1 + N_2)$ symmetry, which in both cases are broken to $U(N_1) \times U(N_2)$. It is easy to see that the expressions for them are related by analytical continuation of $N_2 \rightarrow -N_2$, or analogously a continuation of the 't Hooft coupling $N_2/k \rightarrow -N_2/k$ (which may be attributed to the negative level of the CS coupling of this group in the ABJM theory). We can then go on to study the lens space model and analytically continue to ABJM at the end.

Conveniently, the lens space matrix model has been studied in the past [8, 10, 13, 14]. The planar resolvent is known in closed form and the expressions for its periods are given as power series at special points in moduli space. We review the details of this matrix model and its solution in Sects. 2 and 3.

The matrix model of ABJM theory was derived by localization: it captures in a finite dimensional integral all observables of the full theory which preserve certain supercharges. At the time it was derived in [4], the only such observable (apart for the vacuum) was the 1/6 BPS Wilson loop constructed in [15–17] and 1/2 BPS vortex loop operators [18]. Indeed, the expectation value of the 1/6 BPS Wilson loop can be expressed as an observable in the ABJM matrix model, and by analytical continuation in the lens space model.

Another class of Wilson loop operators, which preserve 1/2 of the supercharges, was constructed in [7] and studied further in [19]. It is the dual of the most symmetric classical string solution in $\text{AdS}_4 \times \mathbb{CP}^3$. This Wilson loop is based on a super-connection in space-time and reduces upon localization to the trace of a supermatrix in the ABJM matrix model [7]. The different 1/2 BPS Wilson loops are classified by arbitrary representations of the supergroup $U(N_1|N_2)$, and the 1/6 BPS ones are classified by a pair

² Though commonly known as ABJ theory, for simplicity we still call the theory with this extra parameter ABJM theory. When specializing to the case of equal rank we refer to it as the “ABJM slice”.

of representations³ of $U(N_1)$ and $U(N_2)$. We will mostly concern ourselves with the 1/2 BPS Wilson loop in the fundamental representation of $U(N_1|N_2)$ and the 1/6 BPS Wilson loop in the fundamental representation of $U(N_1)$. The exception is Sect. 8.2 and Appendix B, where we study the 1/2 BPS Wilson loop in large symmetric and antisymmetric representations. There we also make contact with the vortex loop operators of [18].

Of course, the natural observables in CS theory are the partition function and Wilson loops, so these quantities were also studied earlier in the matrix models of CS (see, for example, [9, 10, 13, 20–22]). This information is encoded in different period integrals on the surface solving the matrix model, as we explain in Sect. 2.2. It turns out that the 1/6 BPS loop is captured by a period integral around one of the two cuts in the planar solution and the 1/2 BPS Wilson loop by a period integral around both cuts, or alternatively, around the point at infinity, and is much easier to calculate [8].

With all this machinery presented in Sects. 2 and 3 in hand, we are ready to calculate, and in Sects. 4, 5 and 6 we study the partition function and Wilson loop observables in the three natural limits of the matrix model. First, in Sects. 4 we look at the orbifold point, which is the weak coupling point of the matrix model and likewise of the physical ABJM theory. The calculations there are straightforward and we present the answers to these quantities. A single term (1/6 BPS loop at 2-loops) was calculated independently directly in the field theory. All other terms are predictions for the higher order perturbative corrections.

Section 5 addresses the strong coupling limit of the theory, where the matrix model should reproduce the semiclassical expansion of these observables in type IIA string theory on $\text{AdS}_4 \times \mathbb{CP}^3$. The expectation value of the Wilson loop was already derived in [8] and matched with a classical string in AdS. We first generalize the strong coupling expansion for the case of $N_1 \neq N_2$, which corresponds to turning on a B-field in the AdS dual. This version of the theory was studied in [11] and a more precise analysis of the dictionary, capturing shifts in the charges, was presented in [23, 24]. Interestingly, it turns out that the matrix model knows about these shifted charges, and the strong coupling parameter turns out to be exactly the one calculated in [24], rather than the naive coupling.

In the same section we present also the calculation of the free energy in the matrix model. The result is proportional to $N^2/\sqrt{\lambda}$ (or a slight generalization for $N_1 \neq N_2$). This scales at large N like $N^{3/2}$, which is indeed the M-theory prediction for the number of degrees of freedom on N coincident M2-branes [25]. Comparing with a supergravity calculation, we find precise agreement with the classical action of $\text{AdS}_4 \times \mathbb{CP}^3$. This is the first derivation of this large N scaling in the field theory side. The matrix model also provides an infinite series of instanton/anti-instanton corrections to both the partition function and to the Wilson loop expectation value, which we interpret as fundamental strings wrapping the \mathbb{CP}^1 inside \mathbb{CP}^3 .

We then turn to a third limit of the theory, when one of the gauge couplings is perturbative and the other one not. In the strict limit the ABJM theory reduces to topological CS and in the matrix model one cut is removed. We show how to perform explicit calculations in this regime both from the planar solution of the matrix model and directly by performing matrix integrals. In both approaches one can see the full lens space matrix model arising as a (rather complicated) observable in topological CS theory on \mathbb{S}^3 . We

³ Special combinations of representations of $U(N_1) \times U(N_2)$ are also representations of $U(N_1|N_2)$, and in this case the 1/6 BPS and 1/2 BPS loops will have the same expression in the matrix model and the same VEVs. The proof of localization for the 1/2 BPS loop [7] relied on this equivalence.

speculate on possible tools of calculating directly in ABJM theory in this limit, where integrating out the bi-fundamental matter fields leads to correlation functions of Wilson loops in CS theory. We demonstrate the idea in the case of the $1/6$ BPS Wilson loop, which has a relatively simple perturbative expansion. This limit of the spin-chain of ABJM theory was considered in [26], and a similar system in four dimensions was studied in [27].

The brave souls that will make it to Sects. 7 and 8 will find some new results on the non-planar corrections to the matrix model, and hence to ABJM theory. In Sect. 7 we show that the full $1/N$ expansion of the free energy on \mathbb{S}^3 is completely determined by a recursive procedure based on direct integration [28, 29] of the holomorphic anomaly equations [30]. The ability to determine the full expansion is closely related to the integrability of topological string theory on toric Calabi–Yau threefolds (as discussed in for example [31]). By the AdS/CFT correspondence, the $1/N$ expansion obtained in this way determines the partition function of type IIA theory on the $\text{AdS}_4 \times \mathbb{CP}^3$ background at all genera. This result is reminiscent of the “old” matrix models for non-critical strings, where a double-scaled $1/N$ expansion, encoded in an integrable system, captures the all-genus partition function of a string theory. The recursive procedure for the computation of the $1/N$ expansion is quite efficient in practice, and one can perform explicit computations at high genus. This allows us to study the large genus behavior of the $1/N$ corrections, and we check that they display the factorial growth $\sim (2g)!$ typical of string perturbation theory [32]. A careful examination of the coefficients suggests that this $1/N$ expansion is Borel summable.

In Sect. 8 we present the genus one correction to the Wilson loop and expand it at both weak and strong coupling. Another topic covered there is that of “giant Wilson loops” [33–35], where in the supergravity dual (at least in $\text{AdS}_5 \times \mathbb{S}^5$) a fundamental string is replaced by a D-brane. This happens for Wilson loops in representations of dimension comparable to N . We calculate the corresponding object in the matrix model and compare it to the vortex loop operators of [18].

One point we have not touched upon is the connection to topological strings. Since CS and the matrix model are related to topological strings, we expect there to be a direct connection between ABJM theory and a topological string theory. All the quantities captured by the matrix model should exist also in a topologically twisted version of ABJM theory, possibly along the lines of [36].

2. The ABJM Matrix Model and Wilson Loops

2.1. The matrix model and its planar limit. The ABJM matrix model, obtained in [4], gives an explicit integral expression for the partition function of the ABJM theory on \mathbb{S}^3 , as well as for Wilson loop VEVs. This matrix model is defined by the partition function

$$\begin{aligned} Z_{\text{ABJM}}(N_1, N_2, g_s) &= \frac{i^{-\frac{1}{2}(N_1^2 - N_2^2)}}{N_1! N_2!} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{dv_j}{2\pi} \frac{\prod_{i < j} \left(2 \sinh \left(\frac{\mu_i - \mu_j}{2}\right)\right)^2 \left(2 \sinh \left(\frac{v_i - v_j}{2}\right)\right)^2}{\prod_{i,j} \left(2 \cosh \left(\frac{\mu_i - v_j}{2}\right)\right)^2} \\ &\quad \times e^{-\frac{1}{2g_s} \left(\sum_i \mu_i^2 - \sum_j v_j^2\right)}, \end{aligned} \quad (2.1)$$

where the coupling g_s is related to the Chern–Simons coupling k of the ABJM theory as

$$g_s = \frac{2\pi i}{k}. \quad (2.2)$$

In writing this matrix integral we have been very careful with its precise overall normalization, since one of our goals in the present paper is to compute the free energy on the sphere at strong coupling. The calculation of [4] captures the full k dependence of the partition function, but we have to fix an overall k -independent normalization. This is done in two steps. First, we require that the above matrix integral reduces to the partition function for Chern–Simons theory on \mathbb{S}^3 when $N_1 = 0$ or $N_2 = 0$ (in a specific framing of \mathbb{S}^3). Once this is done, there is still a k -independent normalization factor which appears as a constant coefficient multiplying the cosh in the denominator. This term was not fixed in [4], but it can be easily obtained from the formulae they presented. This calculation can be found in Appendix A, and leads to the matrix integral (2.1).

The ABJM matrix model is closely related to the $L(2, 1)$ lens space matrix model introduced in [9, 10]. This matrix model is defined by the partition function

$$\begin{aligned} Z_{L(2,1)}(N_1, N_2, g_s) &= \frac{i^{-\frac{1}{2}(N_1^2+N_2^2)}}{N_1!N_2!} \int \prod_{i=1}^{N_1} \frac{d\mu_i}{2\pi} \prod_{j=1}^{N_2} \frac{dv_j}{2\pi} \prod_{i<j} \left(2 \sinh \left(\frac{\mu_i - \mu_j}{2} \right) \right)^2 \left(2 \sinh \left(\frac{v_i - v_j}{2} \right) \right)^2 \\ &\times \prod_{i,j} \left(2 \cosh \left(\frac{\mu_i - v_j}{2} \right) \right)^2 e^{-\frac{1}{2g_s}(\sum_i \mu_i^2 + \sum_j v_j^2)}. \end{aligned} \quad (2.3)$$

The relation between the partition functions is simply [8]

$$Z_{\text{ABJM}}(N_1, N_2, g_s) = Z_{L(2,1)}(N_1, -N_2, g_s). \quad (2.4)$$

Since the large N expansion of the free energy gives a sequence of analytic functions of N_1, N_2 , once these functions are known in one model, they can be obtained in the other by the trivial change of sign $N_2 \rightarrow -N_2$.

Let us now discuss the large N solution of the lens space matrix model, following [8, 10, 13]. At large N , the two sets of eigenvalues, μ_i, v_j , condense around two cuts. The cut of the μ_i eigenvalues is centered around $z = 0$, while that of the v_i eigenvalues is centered around $z = \pi i$. We will write the cuts as

$$\mathcal{C}_1 = (-A, A), \quad \mathcal{C}_2 = (\pi i - B, \pi i + B), \quad (2.5)$$

in terms of the endpoints A, B . It is also useful to use the exponentiated variable

$$Z = e^z, \quad (2.6)$$

In the Z plane the cuts (2.5) get mapped to

$$(1/a, a), \quad (-1/b, -b), \quad a = e^A, \quad b = e^B, \quad (2.7)$$

which are centered around $Z = 1, Z = -1$, respectively, see Fig. 1. We will use the same notation $\mathcal{C}_{1,2}$ for the cuts in the Z plane. The large N solution is encoded in the total resolvent of the matrix model, $\omega(z)$. It is defined as [13]

$$\begin{aligned} \omega(z) &= g_s \left\langle \text{Tr} \left(\frac{Z + U}{Z - U} \right) \right\rangle \\ &= g_s \left\langle \sum_{i=1}^{N_1} \coth \left(\frac{z - \mu_i}{2} \right) \right\rangle + g_s \left\langle \sum_{j=1}^{N_2} \tanh \left(\frac{z - v_j}{2} \right) \right\rangle, \end{aligned} \quad (2.8)$$

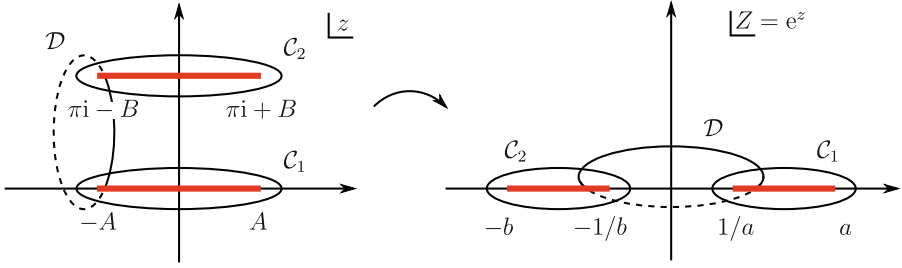


Fig. 1. Cuts in the z -plane and in the Z -plane

where

$$U = \begin{pmatrix} e^{\mu_i} & 0 \\ 0 & -e^{\nu_j} \end{pmatrix}. \quad (2.9)$$

We will denote by $\omega_0(z)$ the planar limit of the resolvent, which was found in explicit form in [13]. It reads,

$$\omega_0(z) = 2 \log \left(\frac{e^{-t/2}}{2} \left[\sqrt{(Z+b)(Z+1/b)} - \sqrt{(Z-a)(Z-1/a)} \right] \right), \quad (2.10)$$

where

$$t = t_1 + t_2 \quad (2.11)$$

is the total 't Hooft parameter. It is useful to introduce the variables⁴

$$\zeta = \frac{1}{2} \left(a + \frac{1}{a} - b - \frac{1}{b} \right), \quad \beta = \frac{1}{4} \left(a + \frac{1}{a} + b + \frac{1}{b} \right). \quad (2.12)$$

β is related to the total 't Hooft parameter through

$$\beta = e^t. \quad (2.13)$$

All the relevant planar quantities can be expressed in terms of period integrals of the one-form $\omega_0(z)dz$. The 't Hooft parameters are given by

$$t_i = \frac{1}{4\pi i} \oint_{C_i} \omega_0(z) dz, \quad i = 1, 2. \quad (2.14)$$

The planar free energy F_0 satisfies the equation

$$\mathcal{I} \equiv \frac{\partial F_0}{\partial t_1} - \frac{\partial F_0}{\partial t_2} - \frac{\pi i t}{2} = -\frac{1}{2} \oint_{\mathcal{D}} \omega_0(z) dz, \quad (2.15)$$

where the \mathcal{D} cycle encloses, in the Z plane, the interval between $-1/b$ and $1/a$ (see Fig. 1).⁵

⁴ The variable β is related to the variable ξ in [8] by $\beta = \xi/2$.

⁵ Likewise one can calculate the second “B-cycle” period, and it will arise when solving the Picard-Fuchs equations at strong coupling in Sect. 3.2.

The derivatives of these periods can be calculated in closed form by adapting a trick from [37]. One finds,

$$\frac{\partial t_{1,2}}{\partial \zeta} = -\frac{1}{4\pi i} \oint_{\mathcal{C}_{1,2}} \frac{dZ}{\sqrt{(Z^2 - \zeta Z + 1)^2 - 4\beta^2 Z^2}} = \pm \frac{\sqrt{ab}}{\pi(1+ab)} K(k), \quad (2.16)$$

and similarly

$$\frac{\partial t_1}{\partial \beta} = -2 \frac{\sqrt{ab}}{\pi(1+ab)} \left(K(k) - \frac{2ab}{1+ab} \Pi(n_1|k) - \frac{2}{1+ab} \Pi(n_2|k) \right), \quad (2.17)$$

where

$$k^2 = 1 - \left(\frac{a+b}{1+ab} \right)^2, \quad n_1 = \frac{1-a^2}{1+ab}, \quad n_2 = \frac{b(a^2-1)}{a(1+ab)}. \quad (2.18)$$

Likewise for the period integral in (2.15) we find

$$\begin{aligned} \frac{\partial \mathcal{I}}{\partial \zeta} &= -2 \frac{\sqrt{ab}}{1+ab} K(k'), \\ \frac{\partial \mathcal{I}}{\partial \beta} &= 4 \frac{\sqrt{ab}}{1+ab} \left(K(k') + \frac{2a(1-b^2)}{(1+ab)(a+b)} (\Pi(n'_1|k') - \Pi(n'_2|k')) \right), \end{aligned} \quad (2.19)$$

where

$$k' = \frac{a+b}{1+ab}, \quad n'_1 = \frac{a+b}{b(1+ab)}, \quad n'_2 = \frac{b(a+b)}{1+ab}. \quad (2.20)$$

We can now use the dictionary between the lens space matrix model and the ABJM matrix model given by (2.2) and (2.4) to get the planar solution of the latter model. In particular, the natural 't Hooft parameters in the ABJM theory

$$\lambda_j = \frac{N_j}{k} \quad (2.21)$$

are obtained from the planar solution of the lens space matrix model by the replacement

$$t_1 = 2\pi i \lambda_1, \quad t_2 = -2\pi i \lambda_2. \quad (2.22)$$

Since in the ABJM theory the couplings $\lambda_{1,2}$ are real, the matrix model couplings $t_{1,2}$ are pure imaginary. Thanks to (2.13) we know that β is of the form

$$\beta = e^{2\pi i(\lambda_1 - \lambda_2)} \quad (2.23)$$

i.e., it must be a phase.

For later convenience we introduce yet another parameterization of the couplings in terms of B and κ ,

$$B = \lambda_1 - \lambda_2 + \frac{1}{2}, \quad \kappa = e^{-\pi i B} \zeta. \quad (2.24)$$

B is identified as the B-field in the dual type IIA background [24]. Notice that it has a shift by $-1/2$ as compared to the original prescription in [11]. Clearly, all calculations in the matrix model are periodic under $B \rightarrow B + 1$, up to possible monodromies (see (5.14) below). As we shall see later, the parameter κ is real for physical values of $\lambda_{1,2}$.

2.2. Wilson loops. One of the main results of [4] is that the VEV of the 1/6 BPS Wilson loop in ABJM theory, labelled by a representation R or $U(N_1)$, can be obtained by calculating the VEV of the matrix e^{μ_i} in the matrix model (2.1), i.e.,

$$\langle W_R^{1/6} \rangle = g_s \langle \text{Tr}_R (e^{\mu_i}) \rangle_{\text{ABJMMM}}. \quad (2.25)$$

A 1/2 BPS loop $W_{\mathcal{R}}^{1/2}$ was constructed in [7], where \mathcal{R} is a representation of the supergroup $U(N_1|N_2)$. In [7] it was also shown that it localizes to the matrix model correlator in the ABJM matrix model

$$\langle W_{\mathcal{R}}^{1/2} \rangle = g_s \langle \text{Str}_{\mathcal{R}} U \rangle_{\text{ABJMMM}}, \quad (2.26)$$

with the same U as in (2.9). Though at first sight the minus sign on the lower block of U , may look surprising, it can be attributed to the fact that the v_j eigenvalues are shifted by πi from the real line. Due to the relation between the ABJM matrix model and the lens space matrix model, these correlators can be computed in the lens space matrix model as follows:

$$\begin{aligned} \langle W_R^{1/6} \rangle &= g_s \langle \text{Tr}_R (e^{\mu_i}) \rangle_{L(2,1)} \Big|_{N_2 \rightarrow -N_2}, \\ \langle W_{\mathcal{R}}^{1/2} \rangle &= g_s \langle \text{Tr}_{\mathcal{R}} U \rangle_{L(2,1)} \Big|_{N_2 \rightarrow -N_2}, \end{aligned} \quad (2.27)$$

where the super-representation \mathcal{R} is regarded as a representation of $U(N_1 + N_2)$.

To evaluate the Wilson loop one uses the resolvent, or equivalently, the eigenvalue densities

$$\begin{aligned} \rho^{(1)}(Z)dZ &= -\frac{1}{4\pi i t_1} \frac{dZ}{Z} (\omega(Z + i\epsilon) - \omega(Z - i\epsilon)), \quad Z \in \mathcal{C}_1, \\ \rho^{(2)}(Z)dZ &= \frac{1}{4\pi i t_2} \frac{dZ}{Z} (\omega(Z + i\epsilon) - \omega(Z - i\epsilon)), \quad Z \in \mathcal{C}_2, \end{aligned} \quad (2.28)$$

which are each normalized in the planar approximation to unity

$$\int_{\mathcal{C}_i} \rho_0^{(i)} dZ = 1. \quad (2.29)$$

For the 1/6 BPS Wilson loop in the fundamental representation one needs to integrate $e^z = Z$ over the first cut

$$\langle W_{\square}^{1/6} \rangle = t_1 \int_{\mathcal{C}_1} \rho^{(1)}(Z) Z dZ = \oint_{\mathcal{C}_1} \frac{dZ}{4\pi i} \omega(Z). \quad (2.30)$$

The correlator relevant for the 1/2 BPS Wilson loop (again in the fundamental representation) is much easier, since

$$\langle W_{\square}^{1/2} \rangle = t_1 \int_{\mathcal{C}_1} \rho^{(1)}(Z) Z dZ - t_2 \int_{\mathcal{C}_2} \rho^{(2)}(Z) Z dZ = \oint_{\infty} \frac{dZ}{4\pi i} \omega(Z), \quad (2.31)$$

and it can be obtained by expanding $\omega(Z)$ around $Z \rightarrow \infty$.

The comparison to the case of the 1/2 BPS Wilson loop in $\mathcal{N} = 4$ SYM in 4d is straight-forward. In that case the matrix model is Gaussian and the eigenvalue density

in the planar approximation follows Wigner's semi-circle law. Doing the integral with the insertion of e^z gives a modified Bessel function [38],

$$\rho_0(z) = \frac{2}{\pi\lambda} \sqrt{\lambda - z^2} \Rightarrow \langle W_{4d\mathcal{N}=4}^{1/2} \rangle_{\text{planar}} = \int_{-\sqrt{\lambda}}^{\sqrt{\lambda}} \rho_0(z) e^z dz = \frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}). \quad (2.32)$$

For the ABJM matrix model all the expressions are more complicated. Still the derivative with respect to ζ and β of the integral expression for the 1/6 BPS Wilson loop (2.30) can be written in closed form [8], like the integrals (2.16) and (2.17),

$$\begin{aligned} \partial_\zeta \langle W_\square^{1/6} \rangle &= -\frac{1}{\pi} \frac{1}{\sqrt{ab}(1+ab)} (a K(k) - (a+b) \Pi(n|k)), \\ \partial_\beta \langle W_\square^{1/6} \rangle &= -\frac{2}{\pi} \frac{\sqrt{ab}}{a+b} E(k). \end{aligned} \quad (2.33)$$

For the 1/2 BPS Wilson loop of [7] the situation is much simpler and in the planar approximation one needs only the large Z behavior of ω_0 (2.10),

$$\omega_0 = t + \frac{\zeta}{Z} + \frac{\zeta^2 + 2\beta^2 - 2}{2Z^2} + \frac{\zeta(\zeta^2 + 6\beta^2 - 3)}{3Z^3} + \mathcal{O}(Z^{-4}). \quad (2.34)$$

One finds [8]

$$\langle W_\square^{1/2} \rangle_{\text{planar}} = \frac{\zeta}{2}, \quad (2.35)$$

which can then be expanded in different regimes. We will elaborate on the expansion of this expression in the next sections and will also turn to the non-planar corrections to it and to that of the 1/6 BPS loop in Sect. 8.

As a simple generalization, by the replacement $Z \rightarrow Z^l$ on the right hand side of (2.31), the higher order terms in the expansion (2.34) give the expectation values of multiply wrapped 1/2 BPS Wilson loops where $U \rightarrow U^l$ in (2.26). For even winding the sign in the lower block of the matrix U (2.9) is absent. This is consistent with the gauge theory calculation [7], where this sign arose from the requirement of supersymmetry invariance in the presence of the fermionic couplings which are antiperiodic, as should be the case for a singly-wound contractible cycle (see also the discussion in [19]).

The normalization of the Wilson loop as given by (2.30) and (2.31) is not the same as in the 4d $\mathcal{N} = 4$ case (2.32). For the 1/6 BPS loop, the leading term at weak coupling is $t_1 = 2\pi i N_1/k$. This means that the trace in the fundamental is normalized by a factor of g_s . For the 1/2 BPS loop the leading term is $t_1 \pm t_2 = g_s(N_1 \mp N_2)$, where the sign depends on the winding number. We will comment more about this normalization in Sect. 5.2.

3. Moduli Space, Picard–Fuchs Equations and Periods

In this section we present the tools for solving the lens space matrix model using special geometry. We present three special points in the moduli space of the theory and write explicit expressions for the four periods of ω_0 at the vicinity of these points.

The lens space matrix model is equivalent to topological string theory on local $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. The $1/N$ expansions of the free energy and of the 1/2 BPS Wilson loop VEV

are the genus expansions of closed and open topological string amplitudes. The planar content of the theory is encoded in the periods of the mirror geometry described by the family of elliptic curves Σ , which can be written as

$$y = \frac{z_1 x^2 + x + 1 - \sqrt{(1 + x + z_1 x^2)^2 - 4z_2 x^2}}{2}. \quad (3.1)$$

Here, z_1, z_2 parametrize the moduli space of complex structures, which is the mirror to the enlarged Kähler moduli space of local \mathbb{F}_0 . This moduli space has a very rich structure first uncovered in [10] and further studied in, for example, [31, 37] by using the standard techniques of mirror symmetry.

Notice that the mirror geometry (3.1) is closely related to the resolvent $\omega_0(Z)$. Indeed, one finds that $\omega_0(Z) \sim \log y(x)$ provided we identify the variables as

$$x = -Zz_1^{-1/2}, \quad (3.2)$$

and

$$\zeta = \frac{1}{\sqrt{z_1}}, \quad \beta = \frac{\sqrt{z_2}}{z_1}. \quad (3.3)$$

This can also be expressed as (2.24)

$$z_1 = \frac{e^{-2\pi i B}}{\kappa^2}, \quad z_2 = \frac{e^{2\pi i B}}{\kappa^2}. \quad (3.4)$$

Let us now discuss in some detail the moduli space of (3.1), since it will play a fundamental role in the following. It has complex dimension two, corresponding to the two complexified Kähler parameters of local \mathbb{F}_0 . The coordinates z_1, z_2 (or ζ, β) are global coordinates in this moduli space. Another way of parametrizing it is to use the periods of the meromorphic one-form

$$\omega = \log y(x) \frac{dx}{x}. \quad (3.5)$$

As it is well-known, these periods are annihilated by a pair of differential operators called Picard–Fuchs operators. In terms of z_1, z_2 , the operators are

$$\begin{aligned} \mathcal{L}_1 &= z_2(1 - 4z_2)\xi_2^2 - 4z_1^2\xi_1^2 - 8z_1z_2\xi_1\xi_2 - 6z_1\xi_1 + (1 - 6z_2)\xi_2, \\ \mathcal{L}_2 &= z_1(1 - 4z_1)\xi_1^2 - 4z_2^2\xi_2^2 - 8z_1z_2\xi_1\xi_2 - 6z_2\xi_2 + (1 - 6z_1)\xi_1, \end{aligned} \quad (3.6)$$

where

$$\xi_i = \frac{\partial}{\partial z_i}. \quad (3.7)$$

These operators lead to a system of differential equations known as *Picard–Fuchs* (PF) equations. An important property of the moduli space is the existence of special points, generalizing the regular singular points of ODEs on \mathbb{C} . The PF system can be solved around these points, and the solutions give a basis for the periods of the meromorphic one-form. We can use two of the solutions to parametrize the moduli space near a singular point, and the resulting local coordinates, given by periods, are usually called *flat coordinates*.

3.1. Orbifold point, or weak coupling. There are three types of special points in the moduli space. The first one is the *orbifold* point discovered in [10], which is the relevant one in order to make contact with the matrix model. To study this point one has to use the global variables,

$$x_1 = 1 - \frac{z_1}{z_2}, \quad x_2 = \frac{1}{\sqrt{z_2} \left(1 - \frac{z_1}{z_2}\right)}. \quad (3.8)$$

The orbifold point is then defined as $x_1 = x_2 = 0$, and in terms of these variables the Picard–Fuchs system is given by the two operators

$$\begin{aligned} \mathcal{L}_1 &= \frac{1}{4}(8 - 8x_1 + x_1^2)x_2\partial_{x_2} - \frac{1}{4}\left(4 - (2 - x_1)^2x_2^2\right)\partial_{x_2}^2 - x_1(2 - 3x_1 + x_1^2)x_2\partial_{x_1}\partial_{x_2} \\ &\quad - (1 - x_1)x_1^2\partial_{x_1} + (1 - x_1)^2x_1^2\partial_{x_1}^2, \\ \mathcal{L}_2 &= (2 - x_1)x_2\partial_{x_2} - (1 - (1 - x_1)x_2^2)\partial_{x_2}^2 - x_1^2\partial_{x_1} \\ &\quad - 2(1 - x_1)x_1x_2\partial_{x_1}\partial_{x_2} + (1 - x_1)x_1^2\partial_{x_1}^2. \end{aligned} \quad (3.9)$$

A basis of periods near the orbifold point was found in [10]. It reads,

$$\begin{aligned} \sigma_1 &= -\log(1 - x_1), \\ \sigma_2 &= \sum_{m,n} c_{m,n} x_1^m x_2^n, \\ \mathcal{F}_{\sigma_2} &= \sigma_2 \log x_1 + \sum_{m,n} d_{m,n} x_1^m x_2^n, \end{aligned} \quad (3.10)$$

where the coefficients $c_{m,n}$ and $d_{m,n}$ vanish for non-positive n or m as well as for all even n . They satisfy the following recursion relations with the seed values $c_{1,1} = 1$, $d_{1,1} = 0$ and $d_{1,3} = -1/6$:

$$\begin{aligned} c_{m,n} &= \frac{(n+2-2m)^2}{4(m-n)(m-1)} c_{m-1,n}, \\ c_{m,n} &= \frac{(n-2)^2(m-n+2)(m-n+1)}{n(n-1)(2m-n)^2} c_{m,n-2}, \\ d_{m,n} &= \frac{(n+2-2m)^3 d_{m-1,n} + 4(n^2 - n - 2m + 2) c_{m,n}}{4(m-1)(m-n)(n+2-2m)}, \\ d_{m,n} &= \frac{(n-2)^2(m-n+1)(m-n+2)}{n(n-1)(2m-n)^2} d_{m,n-2} \\ &\quad + \left(\frac{1}{m-n+2} + \frac{1}{m-n+1} + \frac{4}{n-2m} \right) c_{m,n}. \end{aligned} \quad (3.11)$$

The 't Hooft parameters of the matrix model are period integrals of the meromorphic one-form, therefore they must be linear combinations of the periods above, and one finds [10]

$$t_1 = \frac{1}{4}(\sigma_1 + \sigma_2), \quad t_2 = \frac{1}{4}(\sigma_1 - \sigma_2). \quad (3.12)$$

An expansion around the orbifold point leads to a regime in which t_1, t_2 are very small. In view of (2.22) this corresponds, in the ABJM model, to the *weakly coupled theory*

$$\lambda_1, \lambda_2 \ll 1. \quad (3.13)$$

The remaining period in (3.10) might be used to compute the genus zero free energy of the matrix model. Using the normalization appropriate for the ABJM matrix model, we find

$$\mathcal{I} = 4 \frac{\partial F_0}{\partial \sigma_2} - \frac{\pi i t}{2} = \frac{1}{2} \mathcal{F}_{\sigma_2} - \log(4) \sigma_2 - \frac{\pi i}{2} \sigma_1. \quad (3.14)$$

3.2. Large radius, or strong coupling. The second point that we will be interested in is the so-called *large radius point* corresponding to $z_1 = z_2 = 0$. This is the point where the Calabi–Yau manifold is in its geometric phase, and the expansion of the genus zero free energy near that point leads to the counting of holomorphic curves with Gromov–Witten invariants. The solutions to the Picard–Fuchs Eqs. (3.6) near this point can be obtained in a systematic way by considering the so-called fundamental period

$$\begin{aligned} \varpi_0(z_1, z_2; \rho_1, \rho_2) \\ = \sum_{k, l \geq 0} \frac{\Gamma(2k + 2l + 2\rho_1 + 2\rho_2) \Gamma(1 + \rho_1)^2 \Gamma(1 + \rho_2)^2}{\Gamma(2\rho_1 + 2\rho_2) \Gamma(1 + k + \rho_1)^2 \Gamma(1 + l + \rho_1)^2} z_1^{k+\rho_1} z_2^{l+\rho_2}. \end{aligned} \quad (3.15)$$

As reviewed in for example [39], a basis of solutions to the PF equations can be obtained by acting on the fundamental period with the following differential operators:

$$D_i^{(1)} = \partial_{\rho_i}, \quad D_i^{(2)} = \frac{1}{2} \kappa_{ijk} \partial_{\rho_j} \partial_{\rho_k}. \quad (3.16)$$

Here κ_{ijk} are the classical triple intersection numbers of the Calabi–Yau. This leads to the periods

$$\begin{aligned} T_i(z_1, z_2) &= -D_i^{(1)} \varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0}, \\ F_i(z_1, z_2) &= D_i^{(2)} \varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0}. \end{aligned} \quad (3.17)$$

These periods should be linearly related to those defined in the matrix model in Eqs. (2.14) and (2.15). We present now some explicit expressions for them that we will use in Sects. 5.1 and 5.3 to solve for these relations (see Eqs. (5.3) and (5.23)).

In general, one normalizes these periods and divides them by the fundamental period evaluated at $\rho_1 = \rho_2 = 0$. But in local mirror symmetry we have [40]

$$\varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} = 1. \quad (3.18)$$

The T_i are single-logarithm solutions, and they are identified in standard mirror symmetry with the complexified Kähler parameters, while the F_i are double-logarithm solutions and they are identified with the derivatives of the large radius genus zero free energy w.r.t. the T_i . In our case, we find the explicit expressions

$$\begin{aligned} -T_1 &= \log z_1 + \omega^{(1)}(z_1, z_2), \\ -T_2 &= \log z_2 + \omega^{(1)}(z_1, z_2), \end{aligned} \quad (3.19)$$

where

$$\begin{aligned}\omega^{(1)}(z_1, z_2) &= 2 \sum_{\substack{k, l \geq 0, \\ (k, l) \neq (0, 0)}} \frac{\Gamma(2k+2l)}{\Gamma(1+k)^2 \Gamma(1+l)^2} z_1^k z_2^l \\ &= 2z_1 + 2z_2 + 3z_1^2 + 12z_1 z_2 + 3z_2^2 + \dots\end{aligned}\quad (3.20)$$

In order to obtain the F_i we have to compute the double derivatives w.r.t. the parameters ρ_1, ρ_2 . We find

$$\begin{aligned}\partial_{\rho_1}^2 \varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} \\ = \log^2 z_1 + 2 \log z_1 \omega^{(1)}(z_1, z_2) + \omega_1^{(2)}(z_1, z_2),\end{aligned}\quad (3.21)$$

where

$$\omega_1^{(2)}(z_1, z_2) = 8 \sum_{\substack{k, l \geq 0, \\ (k, l) \neq (0, 0)}} \frac{\Gamma(2k+2l)}{\Gamma(1+k)^2 \Gamma(1+l)^2} (\psi(2k+2l) - \psi(1+k)) z_1^k z_2^l. \quad (3.22)$$

Similarly,

$$\begin{aligned}\partial_{\rho_2}^2 \varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} \\ = \log^2 z_2 + 2 \log z_2 \omega^{(1)}(z_1, z_2) + \omega_2^{(2)}(z_1, z_2),\end{aligned}\quad (3.23)$$

where

$$\begin{aligned}\omega_2^{(2)}(z_1, z_2) &= 8 \sum_{\substack{k, l \geq 0, \\ (k, l) \neq (0, 0)}} \frac{\Gamma(2k+2l)}{\Gamma(1+k)^2 \Gamma(1+l)^2} (\psi(2k+2l) \\ &\quad - \psi(1+l)) z_1^k z_2^l = \omega_1^{(2)}(z_2, z_1).\end{aligned}\quad (3.24)$$

Finally,

$$\begin{aligned}\partial_{\rho_1} \partial_{\rho_2} \varpi_0(z_1, z_2; \rho_1, \rho_2) \Big|_{\rho_1=\rho_2=0} &= \log z_1 \log z_2 + (\log z_1 + \log z_2) \omega^{(1)}(z_1, z_2) \\ &\quad + \frac{1}{2} \left(\omega_1^{(2)}(z_1, z_2) + \omega_2^{(2)}(z_1, z_2) \right).\end{aligned}\quad (3.25)$$

The double log periods are obtained as linear combinations of the above, by using the explicit expressions for the classical intersection numbers that can be found in for example [31]

$$\kappa_{111} = \frac{1}{4}, \quad \kappa_{112} = -\frac{1}{4}, \quad \kappa_{122} = -\frac{1}{4}, \quad \kappa_{222} = \frac{1}{4}. \quad (3.26)$$

We find:

$$\begin{aligned}
 F_1(z_1, z_2) &= -\frac{1}{8} \left(D_{\rho_1}^2 \omega_0 - 2D_{\rho_1 \rho_2} \omega_0 - D_{\rho_1}^2 \omega_0 \right) \\
 &= -\frac{1}{8} \left(\log^2 z_1 - 2 \log z_1 \log z_2 - \log^2 z_2 \right) \\
 &\quad + \frac{1}{4} \log z_2 \omega^{(1)}(z_1, z_2) + \frac{1}{8} \omega_2^{(2)}(z_1, z_2), \\
 F_2(z_1, z_2) &= -\frac{1}{8} \left(D_{\rho_1}^2 \omega_0 - 2D_{\rho_1 \rho_2} \omega_0 - D_{\rho_1}^2 \omega_0 \right) \\
 &= -\frac{1}{8} \left(-\log^2 z_1 - 2 \log z_1 \log z_2 + \log^2 z_2 \right) \\
 &\quad + \frac{1}{4} \log z_1 \omega^{(1)}(z_1, z_2) + \frac{1}{8} \omega_1^{(2)}(z_1, z_2).
 \end{aligned} \tag{3.27}$$

They satisfy the symmetry property

$$F_1(z_1, z_2) = F_2(z_2, z_1). \tag{3.28}$$

The reason why we are interested in the large radius point is because it describes the structure of the ABJM theory at *strong coupling*. In the region where z_2 is small, x_2 is large and the periods $t_{1,2}$ grow. In general, the expansions of the periods around the special points have a finite radius of convergence, but they can be analytically continued to the other “patches”. Since their analytic continuation satisfies the PF equation, we know for example that the analytic continuation of the orbifold periods to the large radius patch must be linear combinations of the periods at large radius. This provides an easy way to perform the analytic continuation which will be carried out in detail in Sect. 5, where we will verify that indeed the region near the large radius point corresponds to

$$\lambda_1, \lambda_2 \gg 1. \tag{3.29}$$

3.3. Conifold locus. Finally, the third set of special points is the *conifold locus*. This is defined by $\Delta = 0$, where

$$\Delta = 1 - 8(z_1 + z_2) + 16(z_1 - z_2)^2. \tag{3.30}$$

In terms of the variables ζ, β , this locus corresponds to the four lines

$$\zeta = -2\beta \pm 2, \quad \zeta = 2\beta \pm 2. \tag{3.31}$$

The conifold locus is the place where cycles in the geometry collapse to zero size. The first two lines correspond to $a = \pm 1$, *i.e.*, the collapse of the C_1 cycle, while the second set of lines corresponds to $b = \mp 1$, *i.e.*, to the collapse of the C_2 cycle. In principle we can solve the PF system near any point in the conifold locus, but in practice it is useful to focus on the point

$$z_1 = z_2 = \frac{1}{16}, \tag{3.32}$$

which has been studied in [31]. We will call it the symmetric conifold point. Appropriate global coordinates around this point are⁶

$$y_1 = 1 - \frac{z_1}{z_2}, \quad y_2 = 1 - \frac{1}{16z_1}. \quad (3.33)$$

In terms of these coordinates, the PF system reads

$$\begin{aligned} \mathcal{L}_1 &= \partial_{y_2} - 2(1 - y_2)\partial_{y_2}^2 - 8(1 - y_1)^2\partial_{y_1} + 8(1 - y_1)^3\partial_{y_1}^2, \\ \mathcal{L}_2 &= -(7 - 8y_2)\partial_{y_2} + 2(3 - 7y_2 + 4y_2^2)\partial_{y_2}^2 - 8(1 - y_1)\partial_{y_1} \\ &\quad - 16(1 - y_1)(1 - y_2)\partial_{y_1}\partial_{y_2} + 8(1 - y_1)^2\partial_{y_1}^2. \end{aligned} \quad (3.34)$$

Notice that, strictly speaking, the orbifold point does not belong to the conifold locus, once the moduli space is compactified and resolved [10]. A generic point in the conifold locus has then $t_1 = 0$ or $t_2 = 0$, but not both, and expanding around the conifold locus means, in the ABJM theory, an expansion in the region

$$\lambda_1 \ll 1, \quad \lambda_2 \sim 1, \quad (3.35)$$

or in the region with λ_2 exchanged with λ_1 . This regime of the ABJM theory has been considered in [26].

It was observed in [41] that the moduli space of the local \mathbb{F}_0 surface can be mapped to a well-known moduli space, namely the Seiberg–Witten (SW) u -plane [42]. This plane is parametrized by a single complex variable u . The relation between the moduli is

$$u = \frac{1}{2} \left(\beta + \beta^{-1} \right) - \frac{\zeta^2}{8\beta}. \quad (3.36)$$

The three singular points that we have discussed (large radius, orbifold, and symmetric conifold) map to the points $u = \infty, +1, -1$. These are the semiclassical, monopole and dyon points of SW theory. As we will see, they can be identified with interesting points in ABJM theory.

An important set of quantities in the study of moduli spaces of CY threefolds are the three-point couplings or Yukawa couplings, $C_{z_i z_j z_k}$. These are the components of a completely symmetric degree three covariant tensor on the moduli space. When expressed in terms of flat coordinates they give the third derivatives of the genus zero free energy. In terms of the coordinates z_1, z_2 , the Yukawa couplings are given by [10, 31]

$$\begin{aligned} C_{111} &= \frac{(1 - 4z_2)^2 - 16z_1(1 + z_1)}{4z_1^3 \Delta}, \\ C_{112} &= \frac{16z_1^2 - (1 - 4z_2)^2}{4z_1^2 z_2 \Delta}, \\ C_{122} &= \frac{16z_2^2 - (1 - 4z_1)^2}{4z_1 z_2^2 \Delta}, \\ C_{222} &= \frac{(1 - 4z_1)^2 - 16z_2(1 + z_2)}{4z_2^3 \Delta}. \end{aligned} \quad (3.37)$$

⁶ These are slightly different from the ones used in [31].

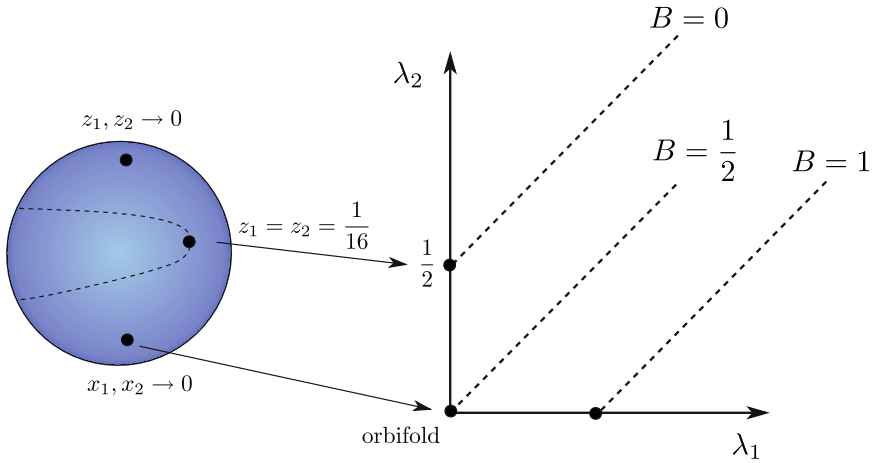


Fig. 2. The moduli space of the ABJM theory, describing the possible values of the 't Hooft couplings $\lambda_{1,2}$, can be parametrized by a real submanifold of the moduli space of local \mathbb{F}_0 , here depicted as a sphere. The orbifold point maps to the origin, while the conifold locus (which is represented by a dashed line) maps to the two axes

3.4. The moduli space of the ABJM theory. The matrix model of ABJM is closely related to the lens space matrix model, and therefore so are also the moduli spaces of the theories. Some of the explicit relations needed for this identification will be presented only in the following sections, but we would still like to present here the main points on the moduli space.

We can think about the *moduli space of the planar ABJM theory* as the space of admissible values of the 't Hooft parameters λ_1, λ_2 . We will assume for simplicity that $k > 0$. The theory with negative values of k can be obtained from this one by a parity transformation. In the gauge theory $\lambda_{1,2}$ must be rational and non-negative (for $k > 0$). Moreover, according to [11], any value of $\lambda_{1,2}$ is admissible as long as

$$|\lambda_1 - \lambda_2| \leq 1. \quad (3.38)$$

This moduli space can be parametrized by the B field and κ , which from the explicit expressions derived below (4.1) and (5.11) has to be real and positive. It can be identified as a real submanifold of the moduli space of local \mathbb{F}_0 . Moreover, we can identify the singular points of this moduli space with natural limits of ABJM theory (see Fig. 2):

1. The *weak coupling regime* $\lambda_{1,2} \rightarrow 0$ corresponds to the orbifold point of the local \mathbb{F}_0 geometry $\kappa = 0$, $B = 1/2$. In terms of type IIA theory, this is also an orbifold geometry with a small radius but a nonzero value for the B field.
2. The *strong coupling regime* $\lambda_{1,2} \rightarrow \infty$ (where also $\kappa \rightarrow \infty$) corresponds to the large radius limit of the local \mathbb{F}_0 geometry.
3. Out of the four lines (3.31) in the (κ, B) plane with $\kappa = \pm 4 \cos \pi B$, which correspond respectively to $a = 1$ and $b = 1$, therefore to $\lambda_1 = 0$ or $\lambda_2 = 0$. Hence, the boundary of the ABJM moduli space given by $\min(\lambda_1, \lambda_2) = 0$ corresponds to

$$\kappa(B) = \begin{cases} -4 \cos \pi B, & B > 1/2 \\ 4 \cos \pi B, & B < 1/2 \end{cases}. \quad (3.39)$$

In particular, the symmetric conifold point $z_1 = z_2 = 1/16$ corresponds to $B = n \in \mathbb{Z}$, $\kappa = \pm 4$. Along the curve (3.39), one of the two gauge groups of the ABJM theory is absent, so the theory reduces to topological CS theory. We examine this regime in Sect. 6.

Given a fixed value of the B field, we can describe the real one-dimensional moduli space of the ABJM theory as a real submanifold of the u -plane of Seiberg–Witten theory, by using (3.36) in the form

$$u = -\cos(2\pi B) + \frac{\kappa^2}{8}. \quad (3.40)$$

Singular points in moduli space become then the well-known singularities of SW theory. For example, when $B = 1/2$, the moduli space, described by $\kappa \in [0, \infty)$, becomes the region $u \in [1, \infty)$. The orbifold point (weakly coupled ABJM theory) maps to the monopole point, while the large radius point (strongly coupled ABJM theory) corresponds to the semi-classical region (see Fig. 3). Notice that the conifold point would map to the dyon point of Seiberg–Witten theory, but this does not belong to the moduli space of ABJM theory with $B = 1/2$. We can however realize it by making an analytic continuation of the 't Hooft coupling to complex values. The dyon point corresponds then to the point $\kappa^2 = -16$, which leads by (5.5) to an imaginary value

$$\lambda = -\frac{2iK}{\pi^2}, \quad (3.41)$$

where K is Catalan's number.

As usual, string dualities lead to a full complexification of the moduli space of 't Hooft parameters. In the case of ABJM theory, the complexified moduli space for the variables $\lambda_{1,2}$ is simply the moduli space of the parameters β, ζ , which is a $\mathbb{Z}_2 \times \mathbb{Z}_2$ covering of the moduli space parametrized by $z_{1,2}$.

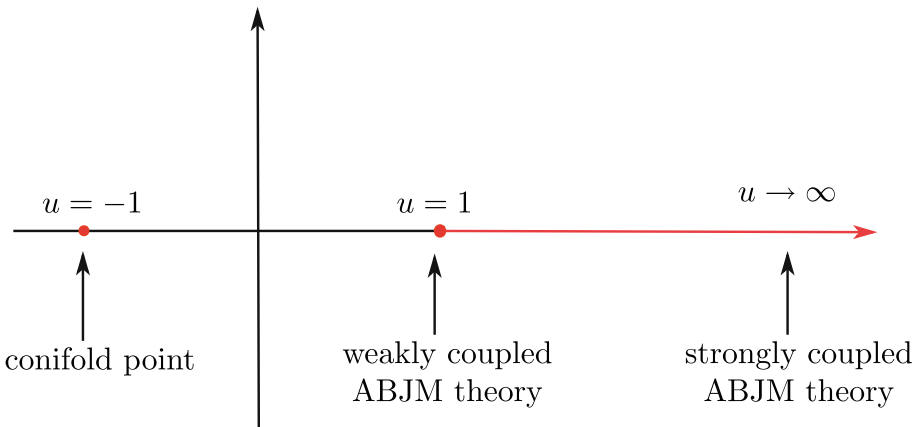


Fig. 3. The moduli space of the ABJM theory for $B = 1/2$ can be mapped to the line $[1, \infty)$ in the u plane of Seiberg–Witten theory, which is here shown in red. The monopole point corresponds to the weakly coupled ABJM theory, while the semiclassical limit corresponds to the strongly coupled theory

4. Weak Coupling

In principle, to study the matrix model at weak coupling one does not need the sophisticated tools presented in the previous section. One can do perturbative calculations directly in the integral expressions (2.1) or (2.3) for the matrix model. A calculation of the 1/6 BPS Wilson loop to three loop order was indeed done in this way in the original paper [4].

Still, the explicit expressions for the periods $\sigma_{1,2}$ (3.10) and their relation to $t_{1,2}$ (3.12) gives a much more efficient way to obtain perturbative, planar expansions. Inverting these relations we find the weak coupling expression for κ (2.24)

$$\begin{aligned} \kappa = & -2i(t_1 - t_2) - \frac{i}{12} \left(t_1^3 + 3t_1^2 t_2 - 3t_1 t_2^2 - t_2^3 \right) \\ & - \frac{i}{960} \left(t_1^5 + 5t_1^4 t_2 - 10t_1^3 t_2^2 + 10t_1^2 t_2^3 - 5t_1 t_2^4 - t_2^5 \right) + \mathcal{O}(t^7). \end{aligned} \quad (4.1)$$

This agrees with the weak coupling expansion of the inverse of the exact mirror map (5.5), obtained in [8].

Using the dictionary relating the 't Hooft couplings (2.22) we immediately get the result for the 1/2 BPS Wilson loop in the planar approximation (2.35),

$$\begin{aligned} \langle W_{\square}^{1/2} \rangle = e^{\pi i B \frac{\kappa}{2}} = e^{\pi i (\lambda_1 - \lambda_2)} 2\pi i (\lambda_1 + \lambda_2) & \left[1 - \frac{\pi^2}{6} \left(\lambda_1^2 - 4\lambda_1 \lambda_2 + \lambda_2^2 \right) \right. \\ & \left. + \frac{\pi^4}{120} \left(\lambda_1^4 - 6\lambda_1^3 \lambda_2 - 4\lambda_1^2 \lambda_2^2 - 6\lambda_1 \lambda_2^3 + \lambda_2^4 \right) + \mathcal{O}(\lambda^6) \right]. \end{aligned} \quad (4.2)$$

In this expression we factored out the term $2\pi i (\lambda_1 + \lambda_2)$, which depends on the overall normalization of the Wilson loop, as mentioned after (2.35). There is also the extra phase factor, which appears also at strong coupling and can be attributed to framing. Note that so far this expansion has not been reproduced directly in the gauge theory, as even the two-loop graphs are quite subtle.

For the 1/6 BPS Wilson loop, using the explicit expression (2.33) and expanding at low orders one finds [8]

$$\begin{aligned} \langle W_{\square}^{1/6} \rangle = e^{\pi i \lambda_1} 2\pi i \lambda_1 & \left(1 - \frac{\pi^2}{6} \lambda_1 (\lambda_1 - 6\lambda_2) \right. \\ & \left. - \frac{\pi^3 i}{2} \lambda_1 \lambda_2^2 + \frac{\pi^4}{120} \lambda_1 \left(\lambda_1^3 - 10\lambda_1^2 \lambda_2 - 20\lambda_2^3 \right) + \mathcal{O}(\lambda^5) \right). \end{aligned} \quad (4.3)$$

Again the exponent is a framing factor and the factor of $2\pi i \lambda_1$ is due to the normalization chosen in (2.30). This expression agrees with the 2-loop calculations in [15–17]. Note that the 3-loop analysis in [17], done for $\lambda_1 = \lambda_2$, misses the next term, due to a projection which essentially removes all terms at odd orders in perturbation theory.

Next we turn to the free energy. Here we notice that the period in (2.15) gives only the derivative of the free energy. Indeed, within the formalism of special geometry developed above, the planar free energy of the matrix model is only determined up to quadratic terms in the 't Hooft couplings. These have to be fixed by direct calculation in the matrix

model

$$F = \frac{N_1^2}{2} \log \left(\frac{2\pi N_1}{k} \right) + \frac{N_2^2}{2} \log \left(\frac{2\pi N_2}{k} \right) - \frac{3}{4} (N_1^2 + N_2^2) - \log(4) N_1 N_2 + \dots \quad (4.4)$$

The last term comes from the normalization of the cosh term in (2.1), while the remaining terms are just the free energies for two Gaussian matrix models with couplings $\pm 2\pi i/k$. Notice that the above free energy has an imaginary piece given by

$$\frac{\pi i}{6k} (N_1 - N_2) ((N_1 - N_2)^2 - 1) \quad (4.5)$$

Using the identification of the periods at weak coupling (3.14) we write down the next term in the perturbative expansion

$$\frac{\pi^2}{72k^2} \left(N_1^4 - 6N_1^3 N_2 + 18N_1^2 N_2^2 - 6N_1 N_2^3 + N_2^4 \right). \quad (4.6)$$

It would be interesting to try to reproduce these expressions directly from studying perturbative ABJM theory on \mathbb{S}^3 .

5. Strong Coupling Expansion and the AdS Dual

We turn now to the strong coupling limit of the matrix model, where we have to find the analytic continuation of the 't Hooft parameters to the strong coupling region, as functions of the global parameters of moduli space. We will see how the shift of the charges discussed in [23,24] emerges naturally from our computation. We will also evaluate the free energy in this regime and compare with the classical action of the vacuum AdS dual, deriving in this way the $N^{3/2}$ behavior of the degrees of freedom.

5.1. Analytic continuation and shifted charges. In order to perform the analytic continuation of the 't Hooft parameters, we use the explicit representation of the periods in terms of integrals given in (2.14) as well as their derivatives (2.16)–(2.17). Let us start by discussing t_1 . We study its behavior at large ζ but fixed β , which is the large radius region. We find

$$\frac{\partial t_1}{\partial \zeta} = \frac{i}{\pi \zeta} \log \left(-\frac{\zeta^2}{\beta} \right) + o(\zeta^{-1}), \quad \frac{\partial t_1}{\partial \beta} = -\frac{i}{2\pi \beta} \left(\log(-\zeta^2) + \pi i \right) + o(1), \quad (5.1)$$

and this gives the leading behavior

$$t_1 = -\frac{i}{2\pi} \left(\log(-\zeta^2) + \pi i \right) \log \frac{\beta}{\zeta} + \dots \quad (5.2)$$

In the physical theory t_1 should be imaginary and β a phase. By examining (5.2), this implies that κ is real. From (3.4) we then see that $z_1 = \bar{z}_2$ and henceforth we label it $z_1 = z$.

We know also that t_1 must be a linear combination of the periods at large radius. Using that $z_1 = 1/\zeta^2$ and $z_2 = (\beta/\zeta)^2$, and comparing (5.2) to the behavior of the periods (3.19) and (3.27), we find

$$\begin{aligned} t_1 &= \frac{i}{2\pi}(F_1 + F_2) - \frac{1}{2}T_2 - \frac{\pi i}{6}, \\ t_2 &= -\frac{i}{2\pi}(F_1 + F_2) + \frac{1}{2}T_1 + \frac{\pi i}{6}. \end{aligned} \quad (5.3)$$

The constants $\pm\pi i/6$ cannot be fixed by using the above information, but they can be fixed by specializing to the ABJM slice $z_1 = z_2$, as we will see in a moment.

A simple calculation leads to the following explicit expression:

$$\begin{aligned} \lambda_1(\kappa, B) &= \frac{1}{2} \left(B^2 - \frac{1}{4} \right) + \frac{1}{24} + \frac{\log^2 \kappa}{2\pi^2} \\ &\quad - \frac{\log \kappa}{2\pi^2} \omega^{(1)}(z, \bar{z}) + \frac{1}{16\pi^2} \left(\omega_1^{(2)} + \omega_2^{(2)} \right)(z, \bar{z}). \end{aligned} \quad (5.4)$$

This expansion is valid in the region $\kappa \rightarrow +\infty$. Notice that it is manifestly real when κ is real and positive.

As a check of the above expression, we can particularize to the ABJM slice $\lambda_1 = \lambda_2 = \lambda$, ($B = 1/2$), which corresponds in the gauge theory, to having identical gauge groups in the two nodes of the quiver, *i.e.*, $N_1 = N_2$. The mirror map for this case was obtained in [8] as

$$\lambda \left(\kappa, B = \frac{1}{2} \right) = \frac{\kappa}{8\pi} {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right). \quad (5.5)$$

The strong coupling expansion of this expression at $\kappa \gg 1$ is

$$\lambda \left(\kappa, B = \frac{1}{2} \right) = \frac{\log^2 \kappa}{2\pi^2} + \frac{1}{24} + \mathcal{O}(\kappa^{-2}), \quad (5.6)$$

in agreement with (5.4). This also fixes the constants in (5.3).

As in [8], the observables of the model are naturally functions of ζ , β (alternatively κ , B), and we have to re-express them in terms of $\lambda_{1,2}$. Equation (5.4) shows that the natural variable at strong coupling is not λ_1 , but rather

$$\hat{\lambda} = \lambda_1 - \frac{1}{2} \left(B^2 - \frac{1}{4} \right) - \frac{1}{24} = \frac{1}{2}(\lambda_1 + \lambda_2) - \frac{1}{2}(\lambda_1 - \lambda_2)^2 - \frac{1}{24}. \quad (5.7)$$

In particular, it is only when expressed in terms of this variable that κ is a periodic function of $\hat{\lambda}$, B .

Remarkably, the above shift is precisely the one found in [24]. In the type IIA realization of the ABJ theory $U(M_2)_k \times U(M_2 + M_4)_{-k}$, where M_2 corresponds to the number of D2 branes and M_4 to the number of D4 branes, the Maxwell charge of the D2 branes is not M_2 , but rather

$$Q_2 = M_2 - \frac{k}{2} \left(B^2 - \frac{1}{4} \right) - \frac{1}{24} \left(k - \frac{1}{k} \right), \quad (5.8)$$

where

$$B = -\frac{M_4}{k} + \frac{1}{2}. \quad (5.9)$$

After dividing by k and taking the large k limit, we recover (5.7) with

$$\hat{\lambda} = \frac{Q_2}{k}. \quad (5.10)$$

The relation between $\hat{\lambda}$ and κ can be inverted at strong coupling, generalizing [8] to $B \neq \frac{1}{2}$, and it is of the form

$$\kappa(\hat{\lambda}, B) = e^{\pi\sqrt{2\hat{\lambda}}} \left(1 + \sum_{\ell \geq 1} c_\ell \left(\frac{1}{\pi\sqrt{2\hat{\lambda}}}, \beta \right) e^{-2\ell\pi\sqrt{2\hat{\lambda}}} \right), \quad (5.11)$$

where

$$c_\ell(x, \beta) = \sum_{k=0}^{2\ell-1} c_k^{(\ell)}(\beta) x^k. \quad (5.12)$$

The coefficients $c_k^{(\ell)}(\beta)$ are Laurent polynomials in β, β^{-1} , of degree ℓ , and symmetric under the exchange $\beta \leftrightarrow \beta^{-1}$. In other words, they can be written as polynomials in $\cos(2\pi m B)$, so they are periodic in B , with period 1. We find, for example,

$$\begin{aligned} c_1(x, \beta) &= -\left(\beta + \beta^{-1}\right) \left(1 - \frac{x}{2}\right), \\ c_2(x, \beta) &= 3 + \frac{x}{8} \left(3\beta^2 - 8 + 3\beta^{-2}\right) \\ &\quad - \frac{3x^2}{8} \left(\beta + \beta^{-1}\right)^2 - \frac{x^3}{8} \left(\beta + \beta^{-1}\right)^2. \end{aligned} \quad (5.13)$$

The fact that $c_\ell(x, \beta)$ are polynomials in x of degree $2\ell - 1$, rather than power series, comes out from an explicit calculation of the first few cases, and we have not established it.

From the explicit expression (5.4) we can implement the symmetries of the model as a function of κ and B (or equivalently, z_1 and z_2). For example, the transformation

$$N_1 \rightarrow 2N_1 + k - N_2, \quad N_2 \rightarrow N_1 \quad (5.14)$$

simply corresponds to periodicity in the B field

$$B \rightarrow B + 1 \quad (5.15)$$

while κ remains unchanged. From the point of view of the $z_{1,2}$ variables, this is simply a monodromy transformation $z_{1,2} \rightarrow e^{\mp 2\pi i} z_{1,2}$. Notice that not all the values of κ lead to admissible values of $\lambda_{1,2}$, since $\min(\lambda_1, \lambda_2) \geq 0$. This means that the boundary of moduli space is the conifold locus (3.39).

5.2. Wilson loops at strong coupling and semi-classical strings. As an application of the explicit expression for κ (5.11), we can use (2.24) to immediately obtain the VEV of the 1/2 BPS Wilson loop (2.35) at strong coupling

$$\langle W_{\square}^{1/2} \rangle_{g=0} = \frac{1}{2} e^{\pi i B} \kappa(\hat{\lambda}, B). \quad (5.16)$$

Note that this is a real function of $\hat{\lambda}$, B , up to the overall phase involving the B field. This is the same phase that appears also in the weak-coupling result (4.2) and arises also in field theory calculations as a framing-dependent term [12, 43, 44]. The matrix model always gives the answer for framing = 1.

The result for the 1/6 BPS Wilson loop is, as usual more complicated, but can still be written in a power series expansion at strong coupling. We quote only the leading strong coupling result for $\lambda_1 = \lambda_2$ [8],

$$\langle W_{\square}^{1/6} \rangle_{g=0} \approx -\frac{\sqrt{2\lambda}}{2} e^{\pi\sqrt{2\lambda}}. \quad (5.17)$$

We would like to comment about the normalization of the operators. As mentioned after (2.35), the normalization chosen there is such that the trace of the identity in the fundamental of $U(N_1)$ gives $t_1 = 2\pi i N_1/k$ and for the fundamental of $U(N_1|N_2)$ (with a minus sign as in (2.9), it gives $t_1 - t_2 = 2\pi i(N_1 + N_2)/k$. In CS theory these normalizations are quite common, but they may be not the most natural ones in the ABJM theory.

An alternative normalization is to divide by this term, such that at weak coupling the expansion of the Wilson loop will be $\langle W \rangle \sim 1 + \dots$. This is the normalization chosen in [8], and hence the slight differences in the preceding equations from that reference. Note, though, that with such a normalization, one would have to divide the doubly-wound 1/2 BPS Wilson loop in the fundamental representation by the super-trace of the identity, which is $2\pi i(N_1 - N_2)/k$ and is singular for $N_1 = N_2$.

There should be a natural choice of normalization that would reproduce the correct normalization of the one-loop partition function of the classical string in $\text{AdS}_4 \times \mathbb{CP}^3$. To this day, though, a fully satisfactory calculation for the analog string in $\text{AdS}_5 \times \mathbb{S}^5$ giving the factor of $\lambda^{-3/4}$ derived from the Gaussian matrix model does not exist. One argument, based on world-sheet arguments was given in [45], but it is not clear why this argument would be modified for ABJM theory. Direct calculations of the determinant [46, 47] were not conclusive. A possible trick to derive it was proposed in [48] by considering a 1/4 BPS generalization of the circular Wilson loop, where three zero modes of the the Wilson loop of [49] are explicitly broken and the integral over them gives this factor. It would be interesting to construct such generalization to the Wilson loop of [7] and see if a similar argument can be derived from that.

Regardless of the overall normalization, one can compare those of the 1/2 BPS loop and the 1/4 BPS loop. Ignoring numerical constants and the framing factor, the ratio is

$$\frac{\langle W_{\square}^{1/6} \rangle_{g=0}}{\langle W_{\square}^{1/2} \rangle_{g=0}} \approx \sqrt{\lambda}, \quad (5.18)$$

which is proportional to the volume of a \mathbb{CP}^1 inside \mathbb{CP}^3 . Indeed, it was argued in [15, 17] that the string description of the 1/6 BPS Wilson loop should be in terms of a string smeared over such a cycle.

5.3. The planar free energy and a derivation of the $N^{3/2}$ behaviour. In this section we study the free energy at strong coupling. We derive the $N^{3/2}$ behavior characteristic of M2 branes [25], and we match the exact coefficient with a gravity calculation in type IIA superstring on $\text{AdS}_4 \times \mathbb{CP}^3$.

The free energy of the matrix model has a large N expansion of the form

$$F = \log Z = \sum_{g=0}^{\infty} g_s^{2g-2} F_g(\lambda_1, \lambda_2). \quad (5.19)$$

This is the way the genus expansion is typically expressed in topological string theory. To compare with the gauge theory and the AdS dual one may choose to rewrite this series as an expansion in powers of $1/N$ by absorbing factors of λ into F_g .

As mentioned in Sect. 4, the formalism of special geometry determines the planar free energy only up to quadratic terms in the 't Hooft couplings, and these have to be fixed from the explicit weak coupling calculation in the matrix model (4.4).

Let us now consider the derivative of the genus zero free energy (2.15), and study its analytic continuation to strong coupling as we have done for t_i at the top of Sect. 5.1. Expanding (2.19) for large κ we find

$$\frac{\partial \mathcal{I}}{\partial \zeta} = -\frac{\pi i}{\zeta} + \mathcal{O}(\zeta^{-2}), \quad \frac{\partial \mathcal{I}}{\partial \beta} = \mathcal{O}(\zeta^{-1}), \quad (5.20)$$

so

$$\mathcal{I} = -\pi i \log \zeta + \mathcal{O}(\zeta^0) = -\pi i \log \kappa + \pi^2 B + \mathcal{O}(\kappa^0, B^0), \quad \kappa \rightarrow \infty. \quad (5.21)$$

From this leading large κ behavior we have that in the ABJM slice

$$\frac{\partial F_0}{\partial \lambda} \approx 2\pi^3 \sqrt{2\lambda}, \quad (5.22)$$

which can be integrated to give the leading term in (5.34) and the match with the supergravity calculation presented below.

But to get the full series of corrections we should proceed more carefully. We know that the result of the continuation should be a linear combination of periods, and comparing to (3.19) we see that we can express the period as

$$\mathcal{I} + \frac{\pi i t}{2} = \frac{\partial F_0}{\partial t_1} - \frac{\partial F_0}{\partial t_2} = -\frac{\pi i}{4} (T_1 + T_2 + 2\pi i). \quad (5.23)$$

The constant term can be fixed by looking at the solution on the ABJM slice $N_1 = N_2$, which can be obtained as follows. Since on the slice we effectively have a one-parameter model, there is only one Yukawa coupling, which we can integrate to obtain F_0 . From (3.37) we easily obtain

$$\partial_\lambda^3 F_0(\lambda) = \frac{1}{4} C_{\lambda\lambda\lambda} \Big|_{\lambda_1 = -\lambda_2} = -\frac{128\pi^6}{\kappa(\kappa^2 + 16)} \frac{1}{K \left(\frac{i\kappa}{4}\right)^3}, \quad (5.24)$$

where the factor of 4 is introduced to match the normalization of the matrix model, and we used that

$$\frac{d\lambda}{d\kappa} = \frac{1}{4\pi^2} K \left(\frac{i\kappa}{4} \right). \quad (5.25)$$

Integrating once, we find

$$\partial_\lambda^2 F_0(\lambda) = 4\pi^3 \frac{K' \left(\frac{i\kappa}{4} \right)}{K \left(\frac{i\kappa}{4} \right)} + a_1, \quad (5.26)$$

where a_1 is an integration constant and we have used the Legendre relation

$$E' K + E K' - K K' = \frac{\pi}{2}. \quad (5.27)$$

A further integration leads to the following expression in terms of a Meijer function:

$$\partial_\lambda F_0(\lambda) = \frac{\kappa}{4} G_{3,3}^{2,3} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| -\frac{\kappa^2}{16} \right) + a_1 \lambda + a_2. \quad (5.28)$$

Comparison with the matrix model free energy at weak coupling (4.4) fixes $a_1 = 4\pi^3 i$, $a_2 = 0$, so we can write

$$\begin{aligned} \partial_\lambda F_0(\lambda) &= \frac{\kappa}{4} G_{3,3}^{2,3} \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \middle| -\frac{\kappa^2}{16} \right) \\ &\quad + \frac{\pi^2 i \kappa}{2} {}_3F_2 \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\frac{\kappa^2}{16} \right). \end{aligned} \quad (5.29)$$

If we integrate this expression with the following choice of integration constant,

$$F_0(\lambda) = \int_0^\lambda d\lambda' \partial_{\lambda'} F_0(\lambda'), \quad (5.30)$$

we obtain the correct weak coupling expansion.

We can now analytically continue the r.h.s. of (5.29) to $\kappa = \infty$, and we obtain

$$\partial_\lambda F_0(\lambda) = 2\pi^2 \log \kappa + \frac{4\pi^2}{\kappa^2} {}_4F_3 \left(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; -\frac{16}{\kappa^2} \right). \quad (5.31)$$

This agrees with (5.23) on the ABJM slice. To see this, one notices that

$$\begin{aligned} \omega^{(1)}(z, z) &= 2 \sum_{n=1}^{\infty} \sum_{k+l=n} \frac{(2k+2l-1)!}{(k!)^2 (l!)^2} z^n = 2 \sum_{n=1}^{\infty} \frac{4^n (2n-1)! \Gamma(n+\frac{1}{2})}{\sqrt{\pi} \Gamma(n+1)^3} z^n \\ &= 4z {}_4F_3 \left(1, 1, \frac{3}{2}, \frac{3}{2}; 2, 2, 2; 16z \right) \end{aligned} \quad (5.32)$$

is precisely the generalized hypergeometric function appearing in (5.31).

We are now ready to discuss the calculation of the planar free energy at strong coupling. We have

$$\partial_{\hat{\lambda}} F_0(\lambda_1, \lambda_2) = 2\pi^2 \log \kappa - \pi^2 \omega^{(1)}(z, \bar{z}). \quad (5.33)$$

After plugging the value of κ in terms of $\hat{\lambda}$ given by the series expansion (5.11), and integrating w.r.t. $\hat{\lambda}$, we obtain

$$F_0(\hat{\lambda}, B) = \frac{4\pi^3 \sqrt{2}}{3} \hat{\lambda}^{3/2} + \sum_{\ell \geq 1} e^{-2\pi \ell \sqrt{2\hat{\lambda}}} f_\ell \left(\frac{1}{\pi \sqrt{2\hat{\lambda}}}, \beta \right) - \frac{2\pi^3 i}{3} \left(B - \frac{1}{2} \right)^3, \quad (5.34)$$

where $f_\ell(x)$ is a polynomial in x of the form

$$f_\ell(x, \beta) = \sum_{k=0}^{2\ell-3} f_k^{(\ell)}(\beta) x^k, \quad \ell \geq 2. \quad (5.35)$$

The coefficients $f_k^{(\ell)}(\beta)$ are Laurent polynomials in β of degree ℓ , and symmetric under the exchange $\beta \leftrightarrow \beta^{-1}$. We have, for the very first cases,

$$\begin{aligned} f_1(x, \beta) &= -\frac{1}{2} (\beta + \beta^{-1}), \\ f_2(x, \beta) &= \frac{1}{16} (\beta^2 + 16 + \beta^{-2}) + \frac{x}{4} (\beta + \beta^{-1})^2. \end{aligned} \quad (5.36)$$

The integration constant in going from (5.33) to (5.34) can be seen to be zero by comparing (5.34) with a numerical calculation of the integral (5.30) at intermediate coupling.

The free energy in the planar approximation is given by rescaling (5.34) by the string coupling $F = g_s^{-2} F_0 + \mathcal{O}(g_s^0)$. This expression displays many interesting features. First, note that on the ABJM slice $N_1 = N_2$ the leading term

$$-\frac{\pi\sqrt{2}}{3} k^2 \hat{\lambda}^{3/2} \quad (5.37)$$

displays the “anomalous” scaling $N^{3/2}$ in the number of degrees of freedom for a theory of M2 branes, as was first derived from a supergravity calculation in [25]. The above calculation is a first principles derivation of this behaviour at strong coupling in the gauge theory. Usually, this behaviour is associated to the thermal free energy on \mathbb{R}^3 , while (5.37) gives rather the free energy of the ABJM theory on \mathbb{S}^3 at strong coupling. However, a supergravity calculation of this free energy also leads to the $N^{3/2}$ behavior. We will show this now, and in particular we will match the numerical coefficient in (5.37).⁷

5.4. Calculation of the free energy in the gravity dual. Consider type IIA theory on $\text{AdS}_4 \times \mathbb{CP}^3$, and let us reduce it to the AdS_4 factor as in for example [50]. The (Euclidean) AdS metric appropriate for a boundary theory on \mathbb{S}^3 is

$$ds^2 = d\rho^2 + \sinh^2 \rho \, d\Omega^2, \quad (5.38)$$

where $d\Omega^2$ is the metric on an \mathbb{S}^3 of unit radius. In this coordinate system, the boundary is at $\rho \rightarrow \infty$. The free energy of the boundary CFT on \mathbb{S}^3 should be given, in the supergravity approximation, by minus the Euclidean gravitational action of the AdS_{n+1} space $-I_{\text{AdS}_{n+1}}$. This action is given by a bulk term, a surface term, and a counterterm at the boundary [51, 52]

$$I_{\text{AdS}_{n+1}} = I_{\text{bulk}} + I_{\text{surf}} + I_{\text{ct}}, \quad (5.39)$$

⁷ We would like to thank Diego Hofman for very useful remarks on this calculation.

with

$$\begin{aligned}
 I_{\text{bulk}} &= -\frac{1}{16\pi G_N} \int_X d^{n+1}x \sqrt{g} (R - 2\Lambda), \\
 I_{\text{surf}} &= -\frac{1}{8\pi G_N} \int_{\partial X} d^n x \sqrt{h} K, \\
 I_{\text{ct}} &= \frac{1}{8\pi G_N} \int_{\partial X} d^n x \sqrt{h} \left[n - 1 + \frac{1}{2(n-2)} \mathcal{R} + \dots \right].
 \end{aligned} \tag{5.40}$$

In these equations, G_N is Newton's constant, and R , K and \mathcal{R} are the scalar curvature of the bulk, the extrinsic curvature of the boundary ∂X , and the scalar curvature of the induced metric h on ∂X , respectively. The counterterm action includes higher order corrections which are not relevant for the case of AdS_4 and will not be considered here [52]. As our boundary ∂X , we will take the hypersurface $\rho = \rho_0$, and at the end of the calculation we must take $\rho_0 \rightarrow \infty$. The counterterms guarantee that the resulting action will be finite.

The bulk action is easy to evaluate and gives

$$I_{\text{bulk}}(\rho_0) = \frac{3}{8\pi G_N} \text{vol}(\text{AdS}_4; \rho_0), \tag{5.41}$$

where

$$\begin{aligned}
 \text{vol}(\text{AdS}_4; \rho_0) &= \text{vol}(\mathbb{S}^3) \int_0^{\rho_0} d\rho (\sinh \rho)^3 \\
 &= 2\pi^2 \left[\frac{1}{12} \cosh(3\rho_0) - \frac{3}{4} \cosh(\rho_0) + \frac{2}{3} \right].
 \end{aligned} \tag{5.42}$$

It is easy to see that the surface term and the counterterms remove the divergences as $\rho_0 \rightarrow \infty$, leaving only the term $4\pi^2/3$ in (5.42), and we find [52]

$$\lim_{\rho_0 \rightarrow \infty} I_{\text{AdS}_4}(\rho_0) = \frac{\pi}{2G_N}. \tag{5.43}$$

If we now use the dictionary relating Newton's constant to the gauge theory data,

$$\frac{1}{G_N} = \frac{2\sqrt{2}}{3} k^2 \hat{\lambda}^{3/2}, \tag{5.44}$$

we find exactly the leading term in (5.34)! Of course, in order to obtain this result we have used the regularization provided by the counterterm integral in (5.40), and one could suspect that the matching depends very much on this regularization. However, this counterterm has been tested (or fixed) in an independent way in the calculations of [51, 52]. In particular, for $n = 4$ it leads to the matching of the Casimir energy of $\mathcal{N} = 4$ SYM on $\mathbb{R} \times \mathbb{S}^3$, and for $n = 3$ it reproduces the standard mass of an AdS_4 -Schwarzschild black hole [51]. Therefore, the above calculation provides a genuine test of the $\text{AdS}_4/\text{CFT}_3$ correspondence.

In Fig. 4 we show the exact result for the planar limit of $\partial_\lambda F_0(\lambda)$ in the case $N_1 = N_2$, as a function of $\lambda = N/k$, and we compare it to the behavior of the supergravity prediction,

$$\partial_\lambda F_0(\lambda) \approx 2\pi^3 \sqrt{2(\lambda - 1/24)}, \quad \lambda \rightarrow \infty. \tag{5.45}$$

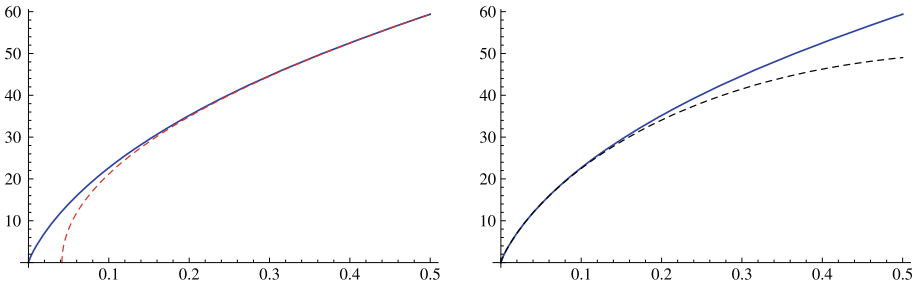


Fig. 4. Comparison of the exact result for $\partial_\lambda F_0(\lambda)$ given in (5.29), plotted as a solid blue line, and the weakly coupled and strongly coupled results. In the figure on the left, the red dashed line is the supergravity result (5.45), while in the figure on the right, the black dashed line is the Gaussian result (5.46)

We see that the strong coupling behavior gets triggered for values of the coupling $\lambda \approx 0.2$. For $\lambda \rightarrow 0$, the behavior of the prepotential is dominated by the Gaussian, weakly coupled result (4.4),

$$\partial_\lambda F_0(\lambda) \approx -8\pi^2 \lambda \left(\log \left(\frac{\pi \lambda}{2} \right) - 1 \right), \quad \lambda \rightarrow 0. \quad (5.46)$$

A second aspect to notice is that the supergravity result (5.34) has corrections which are exponentially suppressed. The exponential is of the form

$$e^{-\ell A(\mathbb{CP}^1)}, \quad (5.47)$$

where

$$A(\mathbb{CP}^1) = 2\pi \sqrt{2\hat{\lambda}} \quad (5.48)$$

is the area of the \mathbb{CP}^1 two-cycle in \mathbb{CP}^3 . Also, notice that each of these exponential corrections multiplies (at each order in $\hat{\lambda}^{-1/2}$) the polynomial $f_k^{(\ell)}(\beta)$ in β, β^{-1} . Therefore, we have contributions schematically of the form

$$\sum_{n_+ + n_- = \ell} c_{n_+, n_-} e^{-n_+(A(\mathbb{CP}^1) + 2\pi i B) - n_-(A(\mathbb{CP}^1) - 2\pi i B)}. \quad (5.49)$$

This is precisely what one should expect for a gas of n_+ instantons and n_- anti-instantons in a σ model on \mathbb{CP}^3 , where the (anti)instantons wrap the \mathbb{CP}^1 cycle. Notice that this kind of corrections are made possible by the non-trivial topology of two cycles in \mathbb{CP}^3 , *i.e.*, by the fact that $b_2(\mathbb{CP}^3) = 1$, and as such they are absent in $\text{AdS}_5 \times \mathbb{S}^5$. Some aspects of these string instantons have been studied in [53]. It would be interesting to test in detail the possible connection between these string instantons and the exponentially suppressed corrections to the planar free energy.

These instanton corrections are also present in the Wilson loop result (5.16), again with an infinite series of corrections. This can be compared with the case of $\mathcal{N} = 4$ SYM in 4d, where the asymptotic large coupling expansion of the Gaussian matrix model (2.32) has a single instanton correction which can be explicitly identified with a second saddle point solution in $\text{AdS}_5 \times S^5$ [48, 54].

Finally, we note that when $N_1 \neq N_2$, the planar free energy (5.34) includes an imaginary term proportional to $(B - 1/2)^3$, which is derived by the weak coupling calculation (4.5). In CS theory such a term is related to framing [12]. It would be very interesting to derive this phase in type IIA string theory.

6. Conifold Expansion

The expansion around the conifold locus corresponds to a region in the moduli space of the ABJM model where one of the gauge groups has finite coupling, while the other one is weakly coupled. In the lens space matrix model this corresponds to one 't Hooft parameter being small, and the other of order 1. In this section we will study this regime from three different points of view: the exact planar solution in terms of periods and Picard–Fuchs equations, the matrix model, and the gauge theory.

6.1. Expansion from the exact planar solution. We can use the exact planar solution to calculate various physical quantities near the conifold locus. For concreteness, we will expand around $t_2 = 0$ but with t_1 arbitrary. The first ingredient we need is an expansion of the global coordinates of moduli space. It turns out that the most convenient method is based on the expressions for the periods (2.14). The locus where $t_2 = 0$ is the line

$$\zeta = 2\beta - 2, \quad (6.1)$$

where the cut $(-b, -1/b)$ collapses to the point $Z = -1$. The derivative of t_2 w.r.t. ζ can then be computed in terms of residues at this point by expanding the expression in (2.16):

$$-\frac{\partial t_2}{\partial \zeta} = \sum_{k \geq 0} \frac{1}{4\pi i} \oint_{-1} dZ \frac{H_k(Z, \beta) (\zeta - 2\beta + 2)^k}{(Z + 1)^{2k+1}}, \quad (6.2)$$

where $H_k(Z, \beta)$ are regular at $Z = -1$. This gives a series for t_2 in powers of $\zeta - 2\beta + 2$,

$$\begin{aligned} -t_2 &= \frac{1}{4\sqrt{\beta}} (\zeta - 2\beta + 2) - \frac{1 - \beta}{128 \beta^{3/2}} (\zeta - 2\beta + 2)^2 \\ &\quad + \frac{9 - 2\beta + 9\beta^2}{12288 \beta^{5/2}} (\zeta - 2\beta + 2)^3 + \mathcal{O}((\zeta - 2\beta + 2)^4), \end{aligned} \quad (6.3)$$

which can be easily inverted to

$$\zeta = 2\beta - 2 - 4\sqrt{\beta} t_2 + \frac{1}{2}(1 - \beta) t_2^2 + \frac{3 + 10\beta + 3\beta^2}{48\sqrt{\beta}} t_2^3 + \mathcal{O}(t_2^4). \quad (6.4)$$

As a nice application of this expansion, we can compute the VEV of the $1/2$ BPS Wilson loop around the conifold point, which is given in (2.35). Using the dictionary (2.23), (2.24), we find

$$\begin{aligned} e^{-\pi i B} \langle W_{\square}^{1/2} \rangle_{g=0} &= 2 \sin(\pi \lambda_1) + 2\pi \lambda_2 (2 - \cos(\pi \lambda_1)) + \pi^2 \lambda_2^2 \sin(\pi \lambda_1) \\ &\quad + \frac{1}{3} \pi^3 \lambda_2^3 \left(1 - 5 \cos(\pi \lambda_1) + 3 \cos^2(\pi \lambda_1) \right) + \mathcal{O}(\lambda_2^4). \end{aligned} \quad (6.5)$$

As $\lambda_2 \rightarrow 0$, we recover the result for a Wilson loop VEV in $U(N_1)$ CS theory. In the conifold expansion we are then regarding the ABJM theory as a perturbation of $U(N_1)$ CS theory at strong coupling.

The above result can be also obtained by solving the Picard–Fuchs equation around a point in the conifold locus. Let us choose for example the symmetric conifold point (3.32), with $B = 1$ and $\kappa = 4$. This corresponds to the point in the conifold locus with

$$\lambda_1 = \frac{1}{2}, \quad \lambda_2 = 0. \quad (6.6)$$

The appropriate global coordinates near this point are (3.33). We find that λ_2 is a period solving the PF system (3.34) and with leading behavior

$$\lambda_2 = -\frac{1}{4\pi} (y_2 + y_1/2) + \mathcal{O}(y^2). \quad (6.7)$$

One finds the expansion

$$\begin{aligned} \lambda_2 = & \frac{\pi}{4} (B-1)^2 - \frac{5\pi^3}{96} (B-1)^4 + \left(\frac{1}{8\pi} - \frac{\pi}{32} (B-1)^2 + \frac{43\pi^3}{1536} (B-1)^4 \right) (\kappa-4) \\ & + \left(-\frac{1}{128\pi} + \frac{9\pi}{1024} (B-1)^2 - \frac{99\pi^3}{8192} (B-1)^4 \right) (\kappa-4)^2 \\ & + \mathcal{O}\left((B-1)^6\right) + \mathcal{O}\left((\kappa-4)^3\right), \end{aligned} \quad (6.8)$$

which is inverted to

$$\begin{aligned} \kappa = & 4 - 2\pi^2 \left(\lambda_1 - \frac{1}{2} \right)^2 + \frac{\pi^4}{6} \left(\lambda_1 - \frac{1}{2} \right)^4 \\ & + \pi \lambda_2 \left(8 + 4\pi \left(\lambda_1 - \frac{1}{2} \right) - \frac{2\pi^3}{3} \left(\lambda_1 - \frac{1}{2} \right)^3 \right) \\ & + \mathcal{O}\left(\lambda_2^2\right) + \mathcal{O}\left((\lambda_1 - 1/2)^5\right). \end{aligned} \quad (6.9)$$

This is indeed the expansion around $\lambda_1 = 1/2$ of (twice) the series in the r.h.s. of (6.5).

Once we know the expansion of the global coordinates, we can consider other quantities in the model, like the genus g free energies. The conifold expansion of $F_g(t_1, t_2)$ has the form

$$F_g(\lambda_1, \lambda_2) = F_g^G(\lambda_2) + \sum_{n \geq 0} F_g^{(n)}(\lambda_1) \lambda_2^n, \quad (6.10)$$

where $F_g^G(\lambda_2)$ is the free energy of the $U(N_2)$ Gaussian matrix model, and each coefficient $F_g^{(n)}(\lambda_1)$ can be obtained as an exact function of λ_1 . Of course,

$$F_g^{(0)}(\lambda_1) = F_g^{\mathbb{S}^3}(\lambda_1) \quad (6.11)$$

is the genus g free energy of the CS theory on \mathbb{S}^3 . When $g = 0$, the expansion (6.10) can be computed from the exact planar solution in various ways. One can for example use the Yukawa couplings (3.37) expanded around the conifold locus in order to compute

the third derivatives of F_0 , or use the modularity properties of the solution discussed in [31, 41]. In any case, for the first few functions one finds the following results:

$$\begin{aligned} F_0^{(1)}(\lambda_1) &= 2\pi i \left(\pi^2 \lambda_1^2 + 2\text{Li}_2 \left(-e^{\pi i \lambda_1} \right) - 2\text{Li}_2 \left(-e^{-\pi i \lambda_1} \right) \right), \\ F_0^{(2)}(\lambda_1) &= -2\pi^3 i \lambda_1 + 8\pi^2 \log \left(\cos \left(\frac{\pi \lambda_1}{2} \right) \right), \\ F_0^{(3)}(\lambda_1) &= \frac{2\pi^3 i}{3} + \frac{\pi^3}{3} (3 \cos(\pi \lambda_1) - 5) \tan \left(\frac{\pi \lambda_1}{2} \right). \end{aligned} \quad (6.12)$$

6.2. Conifold expansion from the matrix model. It is easy to implement the conifold expansion directly in the lens space matrix model. To do that, we notice that it can be written as two interacting Chern–Simons matrix models on \mathbb{S}^3 . We recall that the CS matrix model on \mathbb{S}^3 , first considered in [9], is defined by the partition function

$$Z_{\mathbb{S}^3}(N, g_s) = \frac{1}{N!} \int \prod_{i=1}^N \frac{d\mu_i}{2\pi} \prod_{i < j} \left(2 \sinh \left(\frac{\mu_i - \mu_j}{2} \right) \right)^2 e^{-\frac{1}{2g_s} \sum_i \mu_i^2}. \quad (6.13)$$

This is a one-cut matrix model [21]. It can be obtained from the lens space matrix model when one of the two cuts collapses to zero size. In the Z plane the endpoints of the cut are given by a and a^{-1} , where

$$a = 2e^t - 1 - 2e^{t/2} \sqrt{e^t - 1}. \quad (6.14)$$

Let us consider the following operator in this model:

$$\mathcal{W}(v_j) = 2 \sum_{i,j} \log \left(2 \cosh \left(\frac{\mu_i - v_j}{2} \right) \right). \quad (6.15)$$

The lens space partition function (2.3) can be calculated in two steps. In the first step, we compute

$$Z_1(v_j) = \left\langle e^{\mathcal{W}(v_j)} \right\rangle_{N_1}, \quad (6.16)$$

where the subindex N_1 indicates that this is an unnormalized VEV in the \mathbb{S}^3 CS matrix model with gauge group $U(N_1)$. In a second step, we calculate

$$Z_{L(2,1)} = \langle Z_1(v_j) \rangle_{N_2} \quad (6.17)$$

in the CS matrix model with gauge group $U(N_2)$. To obtain the conifold expansion, we calculate $Z_1(v_j)$ and we expand it in g_s and around $v_j = 0$. Each term in this expansion can be computed exactly as a function of the Kähler parameter t_1 , since the CS matrix model can be solved exactly in the $1/N$ expansion. The resulting double series in g_s and v_j is then regarded as an operator in the CS matrix model with group $U(N_2)$, which we expand around the Gaussian point as in [9, 10], *i.e.*, we expand the sinh measure around $v_j = 0$. The partition function $Z_{L(2,1)}$ is then computed as a VEV in the Gaussian matrix model. This procedure gives a method to compute the expansion (6.10) directly in the matrix model.

To illustrate this procedure, let us calculate $F_0(t_1, t_2)$ at first order in t_2 . In this computation we will denote

$$U_1 = \text{diag}(e^{\mu_i}), \quad U_2 = \text{diag}(e^{\nu_j}). \quad (6.18)$$

The expansion around $\nu_j = 0$ of the operator $\mathcal{W}(\nu_j)$ reads

$$\mathcal{W}(\nu_j) = 2N_2 \sum_{i=1}^{N_1} \log \left[2 \cosh \left(\frac{\mu_i}{2} \right) \right] - \sum_{j=1}^{N_2} \nu_j \sum_{i=1}^{N_1} \tanh \left(\frac{\mu_i}{2} \right) + \mathcal{O}(\nu_j^2). \quad (6.19)$$

The average of the second term in the $U(N_2)$ matrix model vanishes (since it is odd in ν_j), while higher order terms are at least of order t_2^2 . The first term can be written as

$$2 \sum_{i=1}^{N_1} \log \left[2 \cosh \left(\frac{\mu_i}{2} \right) \right] = 2 \text{Tr} \log(1 + U_1) - \sum_{i=1}^{N_1} \mu_i. \quad (6.20)$$

Therefore, in the planar limit and neglecting terms which contribute at order t_2^2 , we have

$$\log Z_1(\nu_j) \approx \frac{2t_2}{g_s} \langle \text{Tr} \log(1 + U_1) \rangle_{N_1}, \quad (6.21)$$

since the second term in (6.20) is odd in μ_i and its VEV vanishes. We then find,

$$F_0(t_1, t_2) = F_0^{\mathbb{S}^3}(t_1) + 2t_2 g_s \langle \text{Tr} \log(1 + U_1) \rangle + \mathcal{O}(t_2^2). \quad (6.22)$$

The VEV in (6.22), which is now normalized, can be computed in terms of the resolvent of the CS matrix model, and similar computations appear in [14, 55] in the context of large N instanton corrections. In fact, it follows from (8.28) and (8.30) that the VEV in (6.22) is given by $-g(-1)$, where $g(Y)$ is computed in (B.2). The final result for the linear correction in t_2 is

$$\frac{\pi^2}{3} + \frac{t_1^2}{2} + \text{Li}_2(e^{-t_1}) - 2\text{Li}_2(e^{-t_1/2}) + 2\text{Li}_2(-e^{-t_1/2}). \quad (6.23)$$

Using dilogarithm identities, this agrees with $\frac{\lambda_2}{t_2} F_0^{(1)}(\lambda_1)$ in (6.12). It is interesting to point out that, in the context of CS theory on the lens space $L(2, 1)$, this function is essentially the action of the large N instanton corresponding to the flat connection

$$U(N) \rightarrow U(N_1) \times U(N_2), \quad N_2 \ll N_1, \quad (6.24)$$

as shown in [14]. In the matrix model, this action is obtained by tunneling N_2 eigenvalues from the first cut to the second cut.

We can also calculate the conifold expansion for the VEV of $1/6$ and $1/2$ BPS Wilson loops directly in the matrix model. We want to compute

$$\langle W_{\square}^{1/6} \rangle = g_s \langle \text{Tr} U_1 \rangle_{L(2,1)}. \quad (6.25)$$

We will again perform this computation in the planar approximation and at linear order in t_2 . At this order we can compute instead the normalized average of the operator

$$\frac{\langle \text{Tr} U_1 e^{\mathcal{W}(\nu_j)} \rangle_{N_1}}{\langle e^{\mathcal{W}(\nu_j)} \rangle_{N_1}} = \langle \text{Tr} U_1 \rangle + \langle \text{Tr} U_1 \mathcal{W}(\nu_j) \rangle^{(c)} + \dots \quad (6.26)$$

in a Gaussian matrix model for the v_j . In the last line, all VEVs are normalized VEVs in the \mathbb{S}^3 CS matrix model. By completing the square of the Gaussian weight we derive

$$\left\langle \text{Tr } U_1 \left(\sum_{i=1}^{N_1} \mu_i \right) \right\rangle = \frac{\partial}{\partial j} \left\langle \text{Tr } U_1 e^{j \sum_{i=1}^{N_1} \mu_i} \right\rangle \Big|_{j=0} = g_s \langle \text{Tr } U_1 \rangle. \quad (6.27)$$

We then find, at this order,

$$\langle W_{\square}^{1/6} \rangle_{g=0} = g_s \langle \text{Tr } U_1 \rangle + t_2 \left(2 \langle \text{Tr } U_1 \text{Tr } \log(1 + U_1) \rangle^{(c)} - g_s \langle \text{Tr } U_1 \rangle \right) + \mathcal{O}(t_2^2). \quad (6.28)$$

The connected correlator

$$\langle \text{Tr } U_1 \text{Tr } \log(1 + U_1) \rangle^{(c)} = - \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \langle \text{Tr } U_1 \text{Tr } U_1^{\ell} \rangle^{(c)} \quad (6.29)$$

can be computed by considering the (partially) integrated two-point function (see for example [56])

$$\int dp W_0(p, q) = - \sum_{n,m} \frac{1}{n p^n q^{m+1}} \langle \text{Tr } U_1^n \text{Tr } U_1^m \rangle^{(c)} \quad (6.30)$$

and extracting the coefficient of q^{-2} . We have,

$$\begin{aligned} & \int dp W_0(p, q) \\ &= \frac{1}{2(p-q)} \left(1 - \sqrt{\frac{(p-a)(p-a^{-1})}{(q-a)(q-a^{-1})}} \right) + \frac{1}{2\sqrt{(q-a)(q-a^{-1})}}, \end{aligned} \quad (6.31)$$

which includes the appropriate integration constant. We find, after changing $p \rightarrow -p$,

$$\begin{aligned} & - \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell p^{\ell}} \langle \text{Tr } U_1 \text{Tr } U_1^{\ell} \rangle^{(c)} \\ &= \frac{1}{4} \left(a + a^{-1} + 2p - 2\sqrt{(p+a)(p+a^{-1})} \right). \end{aligned} \quad (6.32)$$

When $p = 1$ this gives

$$- \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell}}{\ell} \langle \text{Tr } U_1 \text{Tr } U_1^{\ell} \rangle^{(c)} = e^{t_1} - e^{t_1/2}. \quad (6.33)$$

Notice that this is an infinite sum of correlators in the CS matrix model. Since

$$\langle \text{Tr } U_1 \rangle = \frac{e^{t_1} - 1}{g_s}, \quad (6.34)$$

we finally obtain,

$$\begin{aligned}\langle W_{\square}^{1/6} \rangle_{g=0} &= e^{t_1} - 1 + t_2 \left(e^{t_1/2} - 1 \right)^2 + \mathcal{O}(t_2^2) \\ &= e^{t_1/2} \left(2 \sinh \frac{t_1}{2} + t_2 \left(-2 + 2 \cosh \frac{t_1}{2} \right) + \mathcal{O}(t_2^2) \right).\end{aligned}\quad (6.35)$$

Since this is a Wilson loop only in the first group, the framing prefactor depends only on the first 't Hooft coupling.

The $1/2$ BPS Wilson loop is obtained by subtracting

$$\langle \text{Tr } U_2 \rangle_{L(2,1)} = N_2 + \mathcal{O}(t_2^2) = \frac{t_2}{g_s} + \mathcal{O}(t_2^2).\quad (6.36)$$

We find,

$$e^{-(t_1+t_2)/2} \langle W_{\square}^{1/2} \rangle_{g=0} = 2 \sinh \left(\frac{t_1}{2} \right) + t_2 \left(-2 + \cosh \left(\frac{t_1}{2} \right) \right) + \mathcal{O}(t_2^2).\quad (6.37)$$

This is the result (6.5) obtained from the conifold expansion after using the dictionary (2.22).

6.3. On the near Chern–Simons expansion of ABJM theory. In the matrix model the conifold locus corresponds to vanishing of one of the two cuts, where the lens space matrix model can be written as a perturbation around the matrix model for Chern–Simons on \mathbb{S}^3 . Here we want to explore this limit in the original 3-dimensional theory.

In the strict limit we have the theory with $N_2 = 0$ and $N_1 \gg 1$ and arbitrary N_1/k . In this limit all the fields charged under the second gauge group, *i.e.*, its gluons and all the bi-fundamental fermions and scalars are removed. Consequently, ABJM theory simplifies dramatically and reduces to topological $U(N)$ CS. The only observables in the theory in this strict limit are Wilson loops, and they are given by the standard CS answer, which is exact in λ_1 (and $1/N_1$).

One can try to perform a systematic expansion around this point in a perturbative expansion in λ_2 . One keeps $\lambda_2 \ll \lambda_1$, but if desired, can still assume the planar approximation, ignoring also the $1/N_2$ corrections.

It is convenient to draw the Feynman graphs in double-line double-color notation, one color for each group. At the first non-trivial order in λ_2 , only graphs with a single index loop of $U(N_2)$ are included. An arbitrary number of gluons of $U(N_1)$ are allowed. Let us propose the following calculation procedure: First ignore all $U(N_1)$ gluons and enumerate all remaining graphs. They are a very restricted subset, which can be identified very easily.

In order to dress them up with the $U(N_1)$ gluons we write the propagators for the bi-fundamental fields as a path integral over all trajectories in space. As a charged object, these paths will effectively be Wilson loop in $U(N_1)$, which can be calculated exactly in CS theory. Since this theory is topological, the result of adding all the gluons is independent of the path of the bi-fundamental fields. One can then do the usual path integral for these fields and find the regular scalar and fermion propagators.

The statement in the previous paragraph fails in a subtle way. The correlation function of Wilson loop operators in CS theory does not depend on their geometry only as long as their topology — the knotting and linking numbers — are kept fixed. Therefore one

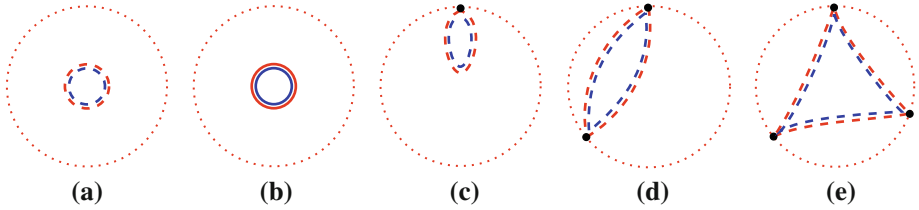


Fig. 5. Several Feynman graphs which may contribute at order λ_2 to the 1/6 BPS Wilson loop with gluons stripped. The big circle is the Wilson loop, dashed lines are bosons and the solid line a fermion, all presented in double-line, double-color notation

has to modify the above statement, and sum over all possible topologies of the paths of the bi-fundamental fields accompanied by the relevant knotted/linked Wilson loops. Unfortunately, we do not have an *a-priori* method of determining the weight that should be assigned to the different topologies.⁸

As an illustration, let us consider the 1/6 BPS Wilson loop (whose Feynman rules are simpler than the 1/2 BPS one) and examine its perturbative expansion about the conifold locus. The Wilson loop is given in our normalization by [15–17]

$$W_{\square}^{1/6} = g_s \text{Tr} \mathcal{P} \exp \int \left(i A_{\mu} \dot{x}^{\mu} + \frac{2\pi}{k} |\dot{x}| M_J^I C_I \bar{C}^J \right) ds. \quad (6.38)$$

x^{μ} parameterizes a circle in \mathbb{R}^3 (or \mathbb{S}^3), A_{μ} are the $U(N_1)$ gluons, C_I and \bar{C}^I are the bi-fundamental scalars and $M_J^I = \text{diag}(1, 1, -1, -1)$ is a matrix in flavor space, which is required to make this object BPS.

At order $\mathcal{O}(\lambda_2^0)$, this is simply a Wilson loop of CS, whose planar expectation value (ignoring framing) is

$$\langle W_{\square}^{1/6} \rangle_{g=0} = 2i \sin \pi \lambda_1 + \mathcal{O}(\lambda_2). \quad (6.39)$$

After stripping away the gluon lines, there are still an infinite number of graphs involving bi-fundamental fields. Examples are shown in Fig. 5. In the example drawn, there is a single scalar or fermion loop. The scalar loop can “touch” the Wilson loop at an arbitrary number of points, due to the scalar bilinear term in (6.38). There are extra graphs which are not drawn, with fermionic tadpoles on the scalar lines, or vice-versa.

By explicit calculations [15–17], all the connected graphs illustrated (Fig. 5c, 5d, 5e) vanish in dimensional regularization. The same can be argued for higher order graphs of this form. Likewise, one would not expect tadpoles to contribute. We are left therefore with the first two disconnected graphs, which become connected once gluon lines are added. Indeed, the only non-vanishing graph that was thus far calculated is the one-loop correction to the gluon propagator (Fig. 5a, 5b with two extra gluons), which accounts for the $\mathcal{O}(\lambda_2 \lambda_1^2)$ term (which with our normalization is 2-loops) in the explicit answer (4.3).⁹

We can compare this to the explicit calculation in the matrix model above. The essential part of the expression for the Wilson loop at order $\mathcal{O}(\lambda_2)$, (6.28) is the connected

⁸ Wilson loops arise out of dressing propagators of matter fields also in [57]. In that case the path is fixed to a collection of light-like segments, due to the singularity in the Minkowski-space propagator.

⁹ This graph has a divergence that can be removed by including the double scalar exchange graph (Fig. 5c). In dimensional regularization the finite part comes only from the gluon graph.

correlator of two Wilson loops. One of them is the original Wilson loop and the other came from expanding the cosh term in the matrix model (6.20), which arises from integrating out the bi-fundamental matter. So this agrees with the identification of the contribution as coming from the bubble graphs. Moreover, what we see in the matrix model is that one should sum over multi-winding of this second Wilson loop, with a weight $1/l$. This corresponds in the physical theory to summing over all possible topologies for the scalar and fermion bubble. As mentioned, we do not know how to derive this factor of $1/l$ from perturbation theory, but it is given to us by the explicit matrix model calculation.

It was noted in [26] that in this limit of ABJM theory the spectrum of local operators also simplifies and the spin-chain hamiltonian becomes short-range. A compelling conjecture for the mysterious function $h(\lambda)$ in that limit was also presented there. It would be interesting to explore this limit further and learn how to do this sum over topologies for other observables.

7. Modular Properties and the Genus Expansion

In this section we provide an efficient, recursive method to compute the $1/N$ corrections to the free energy in the case $N_1 = N_2 = N$. This is based on the modular properties of the solution and the technique of direct integration of the holomorphic anomaly equations. The method determines *a priori* the full $1/N$ expansion. In practice it is quite efficient and it makes possible to calculate the F_g corrections for high genera. This is then used to estimate non-perturbative effects in the large N expansion.

As noted in [41], we can use the relation between the local \mathbb{F}_0 theory and Seiberg–Witten theory to write all the quantities in the model in terms of modular forms. This representation becomes particularly useful when we restrict ourselves to a one-parameter model, as it was shown in a different context in [58]. When $N_1 = N_2$, $\beta = 1$ and the modulus u becomes simply

$$u = 1 + \frac{\kappa^2}{8}. \quad (7.1)$$

In Seiberg–Witten theory, u is related to the modular parameter τ of the Seiberg–Witten curve by

$$u = \frac{\vartheta_4^4 - \vartheta_2^4}{\vartheta_3^4}(\tau) = 1 - 32q^{1/2} + 256q + \dots \quad (7.2)$$

where $q = e^{2\pi i\tau}$. This formula can be inverted to

$$\tau = i \frac{K'(\frac{i\kappa}{4})}{K(\frac{i\kappa}{4})}, \quad (7.3)$$

therefore we see that the modular parameter τ is related to the specific heat of the theory through (5.26). Let us now introduce the quantity

$$\xi = \frac{2}{\vartheta_2^2(\tau)\vartheta_4^4(\tau)}. \quad (7.4)$$

This is proportional to the third derivative of the genus zero free energy, therefore to the Yukawa coupling $C_{\lambda\lambda\lambda}$. More precisely, we have

$$\partial_\lambda^3 F_0(\lambda) = -8\pi^3 i \xi. \quad (7.5)$$

Therefore, the planar content of the theory can be elegantly encoded in terms of modular forms on the Seiberg–Witten curve.

One powerful application of the modular properties of the ABJM theory is the determination of the higher genus corrections to the free energy, $F_g(\lambda)$. These can be obtained in principle from the matrix model (2.1), or equivalently from the formalism of [59] (appropriately modified as in [60,61]). However, as emphasized in for example [28,58,62], this formalism is not very convenient to do calculations at higher genus. One should rather use the fact that the F_g are quasi-modular forms that can be promoted to non-holomorphic modular forms. The resulting non-holomorphic objects satisfy the holomorphic anomaly equations of [30], as shown in [28,63], and these can be in turn solved with the technique of direct integration developed in [28,29,31,58] for local CY manifolds and matrix models.

The basic strategy of direct integration is the following. First, we assume an ansatz for F_g of the form

$$F_g(\tau) = \xi^{2g-2} f_g(\tau), \quad (7.6)$$

where

$$f_g(\tau) = \sum_{k=0}^{3g-3} E_2^k(\tau) c_k^{(g)}(\tau), \quad g \geq 2, \quad (7.7)$$

is an almost modular form of weight $6g - 6$, with respect to a monodromy group $\Gamma \subset SL(2, \mathbb{Z})$. $F_g(\tau)$ can be promoted to a non-holomorphic modular form $F_g(\tau, \bar{\tau})$ by changing

$$E_2(\tau) \rightarrow \widehat{E}_2(\tau, \bar{\tau}) = E_2(\tau) - \frac{3}{\pi \operatorname{Im}(\tau)}. \quad (7.8)$$

The resulting $F_g(\tau, \bar{\tau})$ satisfies the holomorphic anomaly equations of [30], which govern their anti-holomorphic dependence. Since this dependence is contained in $\widehat{E}_2(\tau, \bar{\tau})$, these equations govern the E_2 content of F_g . This means that the coefficients $c_k^{(g)}(\tau)$, which are modular forms of weight $6g - 6 - 2k$, can be obtained recursively for $k > 0$ if one knows the lower F_g . In order to write down the recursive equation, it is useful to introduce a covariant derivative d_ξ taking a form of weight k into a form of weight $k+2$:

$$d_\xi = \partial_\tau + \frac{k}{3} \frac{\partial_\tau \xi}{\xi}. \quad (7.9)$$

Then, the holomorphic anomaly equations lead to

$$\frac{df_g}{dE_2} = -\frac{1}{3} \left\{ d_\xi^2 f_{g-1} + \frac{1}{3} \frac{\partial_\tau \xi}{\xi} d_\xi f_{g-1} + \sum_{r=1}^{g-1} d_\xi f_r d_\xi f_{g-r} \right\}, \quad g \geq 2. \quad (7.10)$$

If $F_{g'}$ are known, with $g' < g$, the above equation determines all the coefficients $c_k^{(g)}(\tau)$ in f_g , with the exception of $c_0^{(g)}(\tau)$, which plays the rôle of an integration constant. This

coefficient is a holomorphic form of weight $6g - 6$ and it is called the *holomorphic ambiguity*.

In order to fix the holomorphic ambiguity we need two pieces of information. The first one concerns its functional dependence. Since $c_0^{(g)}(\tau)$ is a modular form w.r.t. some monodromy subgroup, it belongs to a finitely generated ring. This means that it is determined by a finite number of coefficients, which typically grows with g . The second piece of information comes from boundary conditions at singular points in moduli space. A very powerful boundary condition for matrix models and local Calabi–Yau manifolds is the so-called *gap condition*, discovered in [28] and further used in [31, 58] to fix the holomorphic ambiguity. According to the gap condition, near certain points p_i in moduli space, parametrized by a flat coordinate t_i , the genus g free energy behaves as

$$F_g^{(i)} = \frac{a_g}{t_i^{2g-2}} + \mathcal{O}(1). \quad (7.11)$$

The superscript (i) means that the genus g free energy has to be transformed to the duality frame which is appropriate for the i^{th} singularity, as it is well-known in special geometry. The “gap” refers to the absence of singular terms t^{-k} with $0 < k < 2g - 2$ in the local expansion near $t_i = 0$. The vanishing of these terms provides boundary conditions for $c_0^{(g)}(\tau)$, and in some cases it fixes them completely.

In our case, the relevant ring is that of Γ_2 modular forms which is generated by the theta functions

$$b = \vartheta_2^4(\tau), \quad c = \vartheta_3^4(\tau), \quad d = \vartheta_4^4(\tau). \quad (7.12)$$

Since $c = b + d$, only two of them are independent, and we will choose b and d . Using standard formulae in the theory of modular forms, one finds

$$\frac{\partial_\tau \xi}{\xi} = \frac{b - E_2}{4}, \quad (7.13)$$

as well as

$$d_\xi b = \frac{b^2 + bd}{3}, \quad d_\xi(bd) = \frac{(bd)b}{6}, \quad d_\xi E_2 = \frac{1}{12} \left(-E_2^2 + 2bE_2 - E_4 \right). \quad (7.14)$$

The modular expression for the genus one free energy is known [41] and reads

$$F_1 = -\log \eta(\tau), \quad (7.15)$$

therefore we have

$$d_\xi f_1 = -\frac{E_2}{24}. \quad (7.16)$$

These are all the ingredients needed for the recursion. The holomorphic ambiguity can be written as

$$c_0^{(g)}(\tau) = \sum_{j=0}^{3g-3} \alpha_j^{(g)} b^j d^{3g-3-j} \quad (7.17)$$

and it involves $3g - 2$ unknowns. Let us see how we can fix these by looking at the behavior near the three singular points of moduli space.

At the orbifold point, the F_g are the genus g amplitudes of the super-matrix model (2.1) with $N_1 = N_2$. Their leading behavior near $\lambda = 0$ is governed by two copies of the Gaussian matrix model, therefore they behave as

$$F_g^{(o)}(\lambda) = \frac{B_{2g}}{g(2g-2)}(2\pi i\lambda)^{2-2g} + \mathcal{O}(1). \quad (7.18)$$

This gives $g-1$ conditions, since the ansatz (7.17) for the holomorphic ambiguity only involves even powers of λ .

The symmetric conifold point $z_1 = z_2 = 1/16$ is related to the orbifold point through an S -duality transformation. The appropriate global coordinates near this point are given in (3.33). In the ABJM slice one has

$$y_1 = 0, \quad y_2 = y = 1 - \frac{\xi^2}{16}. \quad (7.19)$$

The following period is a good local, flat coordinate near the symmetric conifold point:

$$t = \sum_{n=0}^{\infty} \frac{a_n}{(n+1)2^{4n}} y^{n+1}, \quad (7.20)$$

where

$$a_n = \frac{1}{\binom{2n}{n}} \sum_{k=0}^n \binom{2k}{k} \binom{4k}{2k} \binom{2n-2k}{n-k} \binom{4n-4k}{2n-2k}. \quad (7.21)$$

It was noticed in [31] that the genus g amplitude at the conifold point behaves like

$$F_g^{(c)}(t) = \frac{B_{2g}}{2g(2g-2)} \left(\frac{t}{2i} \right)^{2-2g} + \mathcal{O}(1). \quad (7.22)$$

This fixes $2g-2$ conditions.

Finally, the large radius point is related to the orbifold point by an STS transformation. The genus g free energy is the generating function of Gromov–Witten invariants of the local \mathbb{F}_0 geometry in the slice $T_1 = T_2 = T$. More precisely, one has

$$F_g^{(\text{GW})}(Q) = (-4)^{g-1} \sum_{d \geq 1} N_{d,g} Q^d, \quad Q = e^{-T}, \quad (7.23)$$

where

$$N_{d,g} = \sum_{d_1+d_2=d} N_{d_1,d_2,g} \quad (7.24)$$

is a sum of Gromov–Witten invariants at genus g , $N_{d_1,d_2,g}$, of local \mathbb{F}_0 (the degrees d_1, d_2 correspond to the two Kähler classes of this geometry). Since (7.23) is a power series in Q with no constant term, we obtain one extra condition, which, together with the $g-1$ conditions from the orbifold point and the $2g-2$ conditions of the conifold point, completely fixes the $3g-2$ unknowns in the holomorphic ambiguity.

Let us see how this works in some detail when $g = 2$. The integration of the holomorphic anomaly equation gives,

$$f_2 = \frac{1}{3} \cdot \frac{1}{24^2} \left(-\frac{5}{3} E_2^3 + 3bE_2^2 - 2E_4E_2 \right) + c_0^{(2)}(\tau), \quad (7.25)$$

where $c_0^{(2)}(\tau)$ is of the form (7.17). The expansion around the orbifold, conifold, and large radius points read, respectively,

$$\begin{aligned} F_2^{(o)}(\lambda) &= \frac{1}{432(2\pi i\lambda)^2} \left(-\frac{11}{3} + 1728\alpha_0^{(2)} \right) + \mathcal{O}(1), \\ F_2^{(c)}(t) &= -\frac{5 + 1296\alpha_3^{(2)}}{1296t^2} + \frac{-1 - 864(12\alpha_2^{(2)} + 15\alpha_3^{(2)})}{10368t} + \mathcal{O}(1), \\ F_2^{(GW)}(Q) &= -\frac{1}{432} \left(\frac{2}{3} + 1728(\alpha_0^{(2)} - \alpha_1^{(2)} + \alpha_2^{(2)} - \alpha_3^{(2)}) \right) + \mathcal{O}(Q). \end{aligned} \quad (7.26)$$

Imposing the conditions (7.18), (7.22) and (7.23) we fix

$$\alpha_0^{(2)} = \frac{1}{25920}, \quad \alpha_1^{(2)} = \frac{7}{17280}, \quad \alpha_2^{(2)} = \frac{1}{3456}, \quad \alpha_3^{(2)} = \frac{1}{3240}. \quad (7.27)$$

We finally obtain

$$\begin{aligned} F_2^{(o)} &= \frac{1}{432bd^2} \left(-\frac{5}{3} E_2^3 + 3bE_2^2 - 2E_4E_2 \right) \\ &\quad + \frac{16b^3 + 15db^2 + 21d^2b + 2d^3}{12960bd^2}. \end{aligned} \quad (7.28)$$

Since τ depends on λ through (7.3) and (5.5), this gives the exact expression for the genus two free energy on \mathbb{S}^3 in the ABJM model, for any value of the 't Hooft coupling.

Notice that the modular ring appearing here and parametrizing the holomorphic ambiguity is different from the one appearing in Seiberg–Witten theory [28, 29] or in the cubic matrix model [58]. This is due to the fact that, although the curves are the same, the meromorphic forms defining the theory are different.

Using this method, we have computed the free energies up to high genus. The strong coupling behavior of $F_g^{(o)}$ is of the form

$$F_g^{(o)}(\lambda) \sim -\lambda^{\frac{3}{2}-g}, \quad \lambda \rightarrow \infty, \quad g \geq 0. \quad (7.29)$$

We have also used these results in order to investigate the large order behavior of the $1/N$ expansion. We have found that

$$\tilde{F}_g(\lambda) = (-1)^{g-1} \left(F_g^{(o)} - \frac{B_{2g}}{g(2g-2)} (2\pi i\lambda)^{2-2g} \right) \quad (7.30)$$

behaves at large g as

$$\tilde{F}_g(\lambda) \sim (2g)! |A(\lambda)|^{-2g} \cos(2g\theta(\lambda) + \delta(\lambda)). \quad (7.31)$$

In this equation, the angle $\theta(\lambda)$ satisfies $\theta(0) = \pi/2$ and $\theta(\lambda) \neq 0$ for all $\lambda > 0$, while $\delta(\lambda)$ is a function of λ (see for example Sect. 2 of [64] for more details on the large

order behavior of the genus expansion). The sign $(-1)^{g-1}$ is included in (7.30) since in the physical ABJM theory the coupling g_s is imaginary. The large order behavior (7.31) indicates that the singularities of the Borel transform of $F_g(\lambda)$ which are closest to the origin are located at $\pm A(\lambda)$, where

$$A(\lambda) = |A(\lambda)| e^{i\theta(\lambda)}. \quad (7.32)$$

Since $\theta(\lambda)$ does not vanish, none of them lies on the positive real axis. This strongly suggests that the $1/N$ expansion of the free energy is Borel summable for any $\lambda > 0$.

The large order behavior of the genus expansion (7.31) is similar to the one found for Chern–Simons theory on \mathbb{S}^3 in [65], and it should be governed by a large N instanton with action $A(\lambda)$. It would be very interesting to identify this instanton and compute $A(\lambda)$ analytically, both in the gauge theory and in the string theory dual. The factorial growth, found here by explicit calculation in the matrix model, agrees with the expected behavior for the genus expansion in string theory [32].

8. More Exact Results on Wilson Loops

In this section we elaborate on the results of [7, 8] and we obtain more exact results on Wilson loops.

8.1. $1/N$ corrections. The higher genus corrections to the VEV of $1/2$ and $1/6$ BPS Wilson loops can be computed in terms of the higher genus corrections to the resolvent of the matrix model. The resolvent has a genus expansion of the form

$$\omega(z) = \sum_{g=0}^{\infty} g_s^{2g} \omega_g(z). \quad (8.1)$$

In the same way, the density of eigenvalues has a large N expansion of the form

$$\rho(z) = \sum_{g=0}^{\infty} g_s^{2g} \rho_g(z), \quad \rho(z) = \rho^{(1)}(z) + \rho^{(2)}(z). \quad (8.2)$$

The $\rho_g^{(i)}(z)$ (with $i = 1, 2$) have their support on the intervals \mathcal{C}_i , and they can be obtained by the discontinuity of ω_g at the cuts as in (2.28).

The genus expansion of the expectation value of the $1/6$ BPS and $1/2$ BPS Wilson loops follows the expressions in (2.30) and (2.31) with the appropriate term in the expansion of $\rho^{(i)}(Z)$ and $\omega(Z)$.

The first step is therefore to compute $\omega_g(p)$. This calculation can be done with the recursive techniques developed in the matrix model literature starting with [56] and culminating with [59]. We will perform an explicit computation for $g = 1$. Calculations for $g \geq 2$ are in principle doable, but they become complicated.

A convenient formula for $\omega_1(p)$ for an algebraic resolvent was found in [66]. To write this formula, we write the discontinuity of the resolvent (also called *spectral curve* in the matrix model literature) as

$$y(p) = M(p) \sqrt{\sigma(p)}, \quad \sigma(p) = (p - x_1)(p - x_2)(p - x_3)(p - x_4). \quad (8.3)$$

$M(p)$ is sometimes called the moment function. Then, one has

$$\omega_1(p) = \frac{4}{\sqrt{\sigma(p)}} \sum_{i=1}^4 \left(\frac{A_i}{(p-x_i)^2} + \frac{B_i}{p-x_i} + C_i \right), \quad (8.4)$$

where

$$\begin{aligned} A_i &= \frac{1}{16} \frac{1}{M(x_i)}, \\ B_i &= -\frac{1}{16} \frac{M'(x_i)}{M^2(x_i)} + \frac{1}{8 M(x_i)} \left(2\alpha_i - \sum_{j \neq i} \frac{1}{x_i - x_j} \right), \\ C_i &= -\frac{1}{48} \frac{1}{M(x_i)} \sum_{j \neq i} \frac{\alpha_j - \alpha_i}{x_j - x_i} \\ &\quad - \frac{1}{16} \frac{M'(x_i)}{M^2(x_i)} \alpha_i + \frac{\alpha_i}{8 M(x_i)} \left(2\alpha_i - \sum_{j \neq i} \frac{1}{x_i - x_j} \right), \end{aligned} \quad (8.5)$$

and the α_i are given by

$$\begin{aligned} \alpha_1 &= \frac{1}{(x_1 - x_2)} \left[1 - \frac{(x_4 - x_2)}{(x_4 - x_1)} \frac{E(k)}{K(k)} \right], \\ \alpha_2 &= \frac{1}{(x_2 - x_1)} \left[1 - \frac{(x_3 - x_1)}{(x_3 - x_2)} \frac{E(k)}{K(k)} \right], \\ \alpha_3 &= \frac{1}{(x_3 - x_4)} \left[1 - \frac{(x_4 - x_2)}{(x_3 - x_2)} \frac{E(k)}{K(k)} \right], \\ \alpha_4 &= \frac{1}{(x_4 - x_3)} \left[1 - \frac{(x_3 - x_1)}{(x_4 - x_1)} \frac{E(k)}{K(k)} \right], \end{aligned} \quad (8.6)$$

where the modulus of the elliptic functions is

$$k^2 = \frac{(x_1 - x_2)(x_3 - x_4)}{(x_1 - x_3)(x_2 - x_4)}. \quad (8.7)$$

These expressions differ from the ones in [66] in a permutation of the roots, as explained in [67]. The overall factor of 4 in (8.4) is due to the fact that our resolvent has a different normalization than the one in [66].

Although the resolvent of the lens space matrix model (2.10) is not algebraic, its discontinuity can be written in the form (8.3) with

$$\sigma(p) = f(p)^2 - 4\beta^2 p^2, \quad f(p) = p^2 - \zeta p + 1 \quad (8.8)$$

and

$$M(p) = \frac{2}{p\sqrt{\sigma(p)}} \tanh^{-1} \frac{\sqrt{\sigma(p)}}{f(p)}. \quad (8.9)$$

This form of the spectral curve is typical of the mirrors of toric geometries [60,61]. The branch points are

$$x_1 = -b, \quad x_2 = -\frac{1}{b}, \quad x_3 = \frac{1}{a}, \quad x_4 = a. \quad (8.10)$$

Using these expressions, it is possible to compute the integral

$$\langle W_{\square}^{1/6} \rangle_{g=1} = \frac{1}{4\pi i} \oint_{C_1} \omega_1(Z) Z dZ \quad (8.11)$$

in closed form, in terms of elliptic functions E , K and the elliptic integral of the third kind $\Pi(n, k)$, with

$$n = \frac{(a^2 - 1)b}{(1 + ab)a}. \quad (8.12)$$

One finds the rather complicated expression

$$\begin{aligned} \langle W_{\square}^{1/6} \rangle_{g=1} = & \frac{1}{12\pi \sqrt{a} b^{3/2} (1 + ab) (a^2 - 1)^2 (b^2 - 1) K} \left[-3(b - 2a + a^2 b) (1 + ab)^4 E^2 \right. \\ & + \left[a(1 + a^4) - b + a^2(4 + 4a^2 - a^4)b - 4a(1 - 3a^2 + a^4)b^2 \right. \\ & \left. - a^2(1 + a^2)b^3(2 + b^2) + a(1 - 8a^2 + a^4)b^4 \right] K^2 \\ & + \left(b^3(1 + 6a^2 + a^4) + 4a(1 + a^2)(b^2 - 1) \right. \\ & \left. + b(3 - 14a^2 + 3a^4) \right) (1 + ab)^2 E K \Big] \\ & + \frac{(ab - 1)(a^2 - b^2)}{12\pi (ab)^{3/2} (1 + ab) k^4 K^2} \left[-6E^2 \right. \\ & \left. + 4(2 - k^2) E K - (2 - 2k^2 + k^4) K^2 \right] \Pi. \end{aligned} \quad (8.13)$$

To check this formula, we expand it around the weakly coupled point $\lambda_1 = \lambda_2 = 0$. After using the inverse mirror map given by (4.1) we find

$$\begin{aligned} \langle W_{\square}^{1/6} \rangle_{g=1} = & -\frac{\pi i}{12} \lambda_1 + \frac{\pi^2}{12} \lambda_1^2 + \frac{\pi^2}{4} \lambda_1 \lambda_2 + \frac{\pi^3 i}{18} \lambda_1^3 + \frac{\pi^3 i}{24} \lambda_1^2 \lambda_2 - \frac{\pi^3 i}{4} \lambda_1 \lambda_2^2 \\ & - \frac{\pi^4}{36} \lambda_1^4 + \frac{\pi^4}{24} \lambda_1^3 \lambda_2 + \frac{5\pi^4}{24} \lambda_1^2 \lambda_2^2 - \frac{\pi^4}{6} \lambda_1 \lambda_2^3 + \mathcal{O}(\lambda^5). \end{aligned} \quad (8.14)$$

We can test this expansion with a perturbative calculation in the ABJM matrix model. At order $\mathcal{O}(g_s^4)$ we have found,

$$\begin{aligned} \frac{e^{-g_s N_1/2}}{2\pi i \lambda_1} \langle W_{\square}^{1/6} \rangle = & 1 - \left(\frac{1}{24} N_1^2 - \frac{1}{4} N_1 N_2 + \frac{1}{24} \right) g_s^2 + \left(\frac{1}{16} N_1^2 N_2 - \frac{1}{16} N_2 \right) g_s^3 \\ & + \left(\frac{3}{5760} N_1^4 - \frac{10}{1920} N_1^3 N_2 - \frac{20}{1920} N_1 N_2^3 \right. \\ & \left. - \frac{10}{5760} N_1^2 + \frac{5}{192} N_1 N_2 + \frac{1}{32} N_2^2 + \frac{7}{5760} \right) g_s^4 + \dots \end{aligned} \quad (8.15)$$

It is straightforward to see that this agrees with (8.14).

The $1/N$ correction to the $1/2$ BPS Wilson loop is much easier to obtain, since it can be computed as a residue at infinity. We have that

$$\omega_1(Z) = \frac{4}{Z^2} \sum_{i=1}^4 C_i + \mathcal{O}(Z^{-3}), \quad (8.16)$$

where the C_i are given in (8.5). We find, at weak coupling,

$$\begin{aligned} \langle W_{\square}^{1/2} \rangle_{g=1} = & -\frac{\pi i}{12}(\lambda_1 + \lambda_2) + \frac{\pi^2}{12}(\lambda_1^2 - \lambda_2^2) + \frac{\pi^3 i}{18}(\lambda_1^3 + \lambda_2^3) - \frac{5\pi^3 i}{24}\lambda_1\lambda_2(\lambda_1 - \lambda_2) \\ & - \frac{\pi^4}{36}(\lambda_1^4 - \lambda_2^4) + \frac{5\pi^4}{24}\lambda_1\lambda_2(\lambda_1^2 - \lambda_2^2) + \mathcal{O}(\lambda^5). \end{aligned} \quad (8.17)$$

At strong coupling we find (we consider for simplicity the ABJM slice)

$$\langle W_{\square}^{1/2} \rangle_{g=1} = \frac{1}{24i} \frac{3 + 2 \log^2 \kappa - 4 \log \kappa}{\log^2 \kappa} \kappa + \mathcal{O}(1). \quad (8.18)$$

The leading exponent is exactly as at genus zero (5.16), representing the same minimal surface with an extra degenerate handle attached. Its effect is to modify the one-loop determinant, which (with our normalization and ignoring instantons) can be written as

$$\langle W_{\square}^{1/2} \rangle_{g=1} = -i \left(\frac{1}{12} - \frac{1}{6\pi\sqrt{2\lambda}} + \frac{1}{16\pi^2\lambda} \right) e^{\pi\sqrt{2\lambda}}, \quad \lambda \rightarrow \infty. \quad (8.19)$$

8.2. Giant Wilson loops. It has been argued in [33–35, 68] that a D-brane probe in $\text{AdS}_5 \times \mathbb{S}^5$ represents an insertion of a Wilson loop in the dual 4d $\mathcal{N} = 4$ SYM with a large symmetric or antisymmetric representation (in the case of D3 branes and D5 branes, respectively). These “giant Wilson loops” are characterized by a representation with n boxes, and one considers the limit

$$n, N \rightarrow \infty, \quad \frac{n}{N} \text{ fixed}. \quad (8.20)$$

In terms of the Gaussian matrix model of the Wilson loops in that theory, the giant Wilson loop in the symmetric representation is represented by an additional eigenvalue outside the cut and the antisymmetric representation by a “hole” in the original cut.

Let us review now the known D-brane solutions which could be relevant for ABJM theory. The usual $1/2$ BPS Wilson loop in the fundamental representation is described by a string with world-volume $\text{AdS}_2 \subset \text{AdS}_4$. In M-theory it is an M2-brane wrapping also the orbifold cycle on $\mathbb{S}^7/\mathbb{Z}_k$. When considering $k/2$ coincident M2-branes (or k , when it is odd) the M2-brane solution develops an extra branch, where the circle becomes a linear combination of the orbifold direction and a contractible circle in AdS_4 [69]. In type IIA these configurations are D2-branes with world-volume $\text{AdS}_2 \times \mathbb{S}^1 \subset \text{AdS}_4$, where the radius of the \mathbb{S}^1 is a free modulus. From the M-theory point of view these are continuous deformations of the system of $k/2$ coincident M2-branes describing a Wilson loop in a $k/2$ dimensional representation. In the field theory they are the vortex loop operators of [18], which have a description as semi-classical field configurations and carry the same charge as $k/2$ Wilson loops.

These solutions have further moduli associated to rotations away from the orbifold cycle inside $\mathbb{S}^7/\mathbb{Z}_k$. Such M2-brane configurations preserve 8 supercharges (1/3 BPS) [15, 18].

There is also a known family of D6-brane solutions which were argued in [15] to represent the 1/6 BPS Wilson loops in anti-symmetric representations. The action for this D-branes is (for $N_1 = N_2$)

$$S_{D6} = -\pi\sqrt{2\lambda} \frac{n(N-n)}{N}, \quad (8.21)$$

which matches that of n strings for small n and has the $n \rightarrow N - n$ symmetry of the antisymmetric representation. In the matrix model these D6-branes should correspond to creating a “hole” in one of the two cuts, splitting it in two.

We turn now to the lens space matrix model and try to find the appropriate description for these objects, and in particular the 1/2 BPS vortex loop operators. As pointed out in [70], the calculation of Wilson loops in the matrix model in this limit can be done in a saddle-point approximation. We will now reformulate the arguments of [70] and adapt them to the lens space matrix model.

We will focus on the case of 1/2 BPS Wilson loops, where we want to calculate

$$W_n^\eta = \langle \text{Tr}_{\mathcal{R}_n^\eta} U \rangle, \quad \eta = \pm 1, \quad (8.22)$$

where U is the same matrix as in (2.9) and $\mathcal{R}_n^{\pm 1} = S_n, A_n$ are respectively the totally symmetric and the totally antisymmetric representations of $U(N_1 + N_2)$ with n boxes. It will turn out that the relevant limit in this theory is slightly different from (8.20) and is given by fixing

$$\nu = \eta \frac{n}{k} = \frac{\eta g_s n}{2\pi i}. \quad (8.23)$$

Positive ν will correspond to symmetric representations and negative ν to antisymmetric ones. In the 't Hooft limit, for fixed N/k , the two scalings are clearly equivalent.

The calculation of (8.22) is very similar to the calculation of partition functions of n bosons or fermions in the *canonical* ensemble, where n is fixed and large. But at large n , in the thermodynamic limit, this calculation can be done as well in the *grand canonical* ensemble. We then introduce the fugacity z and consider the grand-canonical partition function, using the expression for the determinant as the generating function of the characters

$$\Xi_\eta(z) = \sum_{n \geq 0} z^n W_n^\eta = \langle \det(1 - \eta z U)^{-\eta} \rangle = \left\langle \exp \left(\sum_{\ell \geq 1} \frac{\text{Tr } U^\ell}{\ell} \eta^{\ell-1} z^\ell \right) \right\rangle. \quad (8.24)$$

The average value of n in this ensemble is given by (we remove the average notation here, as is standard in the grand canonical formalism)

$$n = z \frac{\partial}{\partial z} \log \Xi_\eta. \quad (8.25)$$

This is inverted to determine the fugacity as a function of the number of particles

$$z_* = z_*(n), \quad (8.26)$$

and then the original VEV can be calculated, in a saddle point approximation, as

$$W_n^\eta \approx z_*^{-n} \Xi_\eta(z_*) = \left\langle \exp \left(-n \log z_* + \sum_{\ell \geq 1} \frac{\text{Tr } U^\ell}{\ell} \eta^{\ell-1} z_*^\ell \right) \right\rangle. \quad (8.27)$$

For convenience, let us henceforth absorb $Y = \eta z$. It can be seen that, at leading order in large N , the grand-canonical partition function (8.24) is given by disconnected planar graphs. Therefore

$$\Xi_\eta(Y) \approx \exp \left(\frac{\eta}{g_s} g(Y) \right), \quad g(Y) = g_s \sum_{\ell \geq 1} \frac{\langle \text{Tr } U^\ell \rangle_0}{\ell} Y^\ell, \quad (8.28)$$

where the subscript 0 refers to the planar part. We now observe that the function $g(Y)$ is related to the planar resolvent in the lens space matrix model (2.8) and (2.10) by

$$\begin{aligned} Y \frac{\partial}{\partial Y} g(Y) &= \frac{1}{2} \left(\omega_0(Y^{-1}) - t \right) \\ &= -\log \left(\frac{1}{2} \left[\sqrt{(Y+b)(Y+1/b)} + \sqrt{(Y-a)(Y-1/a)} \right] \right). \end{aligned} \quad (8.29)$$

Note that compared to ω_0 in (2.10), the sign between the two square roots is reversed. Integrating this equation we get

$$g(Y) = - \int_0^Y \frac{dY'}{Y'} \log \left(\frac{1}{2} \left[\sqrt{(Y'+b)(Y'+1/b)} + \sqrt{(Y'-a)(Y'-1/a)} \right] \right). \quad (8.30)$$

The initial point of integration is chosen to be $Y = 0$, since around that point the integrand approaches a constant $\zeta/2 + \mathcal{O}(Y)$. This guaranties that for small Y the result of the integration will be proportional to the 1/2 BPS Wilson loop (2.35).

The saddle point Eqs. (8.25) determining the mean value of n is then given by

$$\nu = \frac{1}{2\pi i} Y \frac{\partial}{\partial Y} g(Y), \quad (8.31)$$

i.e., (8.29),

$$e^{-2\pi i \nu} = \frac{1}{2} \left[\sqrt{(Y_*+b)(Y_*+1/b)} + \sqrt{(Y_*-a)(Y_*-1/a)} \right], \quad Y_* = \eta z_*. \quad (8.32)$$

This can be solved explicitly in terms of β , ζ or alternatively in terms of B and κ . The solution reads

$$Y_* = \frac{i\kappa e^{-\pi i(2\nu+B)}}{4 \sin(2\pi(\nu+B))} \left(1 - \sqrt{1 - \frac{16 \sin(2\pi\nu) \sin(2\pi(\nu+B))}{\kappa^2}} \right). \quad (8.33)$$

The choice of sign is such that $Y_* = 0$ when $\nu = 0$. We will write

$$W_n^\eta \approx \exp(A_\eta/g_s), \quad (8.34)$$

where A_η , which is identified with the action of a brane probe in the large N string/M-theory dual, is given by

$$A_\eta = -2\pi i \eta \nu \log(\eta Y_*) + \eta g(Y_*). \quad (8.35)$$

In the original variables, in terms of ω_0 , the integral (8.30) is from infinity to a finite position Y_*^{-1} , and represents the effect of adding a single eigenvalue to the system. This fits with the standard dictionary [71] identifying a brane with a single eigenvalue.

This integral gives an expression for the action of the giant Wilson loop, in the limit (8.20) which is exact as a function of the 't Hooft couplings. The derivatives of this integral with respect to β and ζ can be evaluated in closed form, as in (2.16), in terms of *incomplete* elliptic integrals. The resulting expression can then be studied at the different limits of the ABJM theory as done for other observables in earlier sections.

If we go to the conifold limit, setting $\lambda_2 = 0$, we get an expression for the giant Wilson loop in Chern–Simons theory on \mathbb{S}^3 . In that case there exists an exact expression for the Wilson loop for all n . As we show in Appendix B, the above derivation in this limit indeed reproduces the CS answer.

We will now discuss the expansion of the result for the giant Wilson loop for large κ , since this is the strong coupling limit in which one makes contact with the AdS geometry [33]. In terms of B and κ , the integral (8.30) reads

$$g(Y_*) = - \int_0^{Y_*} \frac{dY'}{Y'} \log \left(\frac{1}{2} \left[\sqrt{(1+Y')^2 - e^{\pi i B} Y' (\kappa - 4i \sin(\pi B))} + \sqrt{(1-Y')^2 - e^{\pi i B} Y' (\kappa + 4i \sin(\pi B))} \right] \right), \quad (8.36)$$

where Y_* is given in (8.33).

Expanding Y_* at leading order at large κ we get

$$Y_* = 2i e^{-\pi i (2\nu+B)} \frac{\sin(2\pi \nu)}{\kappa} + \mathcal{O}(\kappa^{-2}) = \frac{1 - e^{-4\pi i \nu}}{\kappa} e^{-\pi i B} + \mathcal{O}(\kappa^{-2}). \quad (8.37)$$

This suggests rescaling Y in the integral (8.36) by κ , which allows for a systematic expansion in powers of κ^{-1} . At leading order the integral becomes

$$g(Y_*) = - \int_0^{Y_*} \frac{dY'}{Y'} \left(\log \sqrt{1 - e^{\pi i B} \kappa Y'} + \mathcal{O}(\kappa^{-1}) \right). \quad (8.38)$$

This yields

$$g(Y_*) = \frac{1}{2} \text{Li}_2(e^{\pi i B} \kappa Y_*) + \mathcal{O}(\kappa^{-2}) = \frac{1}{2} \text{Li}_2(1 - e^{-4\pi i \nu}) + \mathcal{O}(\kappa^{-2}). \quad (8.39)$$

Another way to get this estimate is to notice that the highest powers of ζ in the series expansion in y of $g(y)$ are captured by

$$g(y) = \frac{1}{2} \text{Li}_2(\zeta y) + \dots. \quad (8.40)$$

Using the dilogarithm identity (B.3) we conclude that the action (8.35), written in terms of the original variable n , is

$$\begin{aligned} \frac{1}{g_s} A_\eta = & n\pi\sqrt{2\hat{\lambda}} + \frac{n\pi i}{2}(2B - 1 + \eta) \\ & + \frac{\eta k}{4\pi i} \left(\frac{\pi^2}{6} - \text{Li}_2\left(e^{-4\pi i n/k}\right) \right) + \mathcal{O}(\hat{\lambda}^{-1/2}, e^{-2\pi\sqrt{2\hat{\lambda}}}). \end{aligned} \quad (8.41)$$

Notice that this formula does not display the exchange symmetry $n \leftrightarrow N - n$ for the antisymmetric case $\eta = -1$. This is because this symmetry is not present for the antisymmetric super-representation, as pointed out in [72].

The leading order in λ in (8.41) is as expected, *i.e.*, n times the action of the fundamental string (and n times an extra framing factor). The non-trivial dependence on ν only appears at subleading order in λ , and therefore will not be visible in the supergravity approximation. As mentioned above, there are no known $1/2$ BPS brane solutions carrying less than $k/2$ units of electric charge other than fundamental strings. So we expect that the above action describes the interaction of these coincident strings.

For n a multiple of $k/2$ (or of k , if it is odd), we see from (8.33) that $Y_* = 0$ and the integral (8.30) is over a full cycle. The argument of the dilogarithm in (8.41) is unity, canceling the $\pi^2/6$ term. Since Y^* passed through one of the cuts \mathcal{C}_1 or \mathcal{C}_2 , it is now on a different sheet, and exactly at the branch point of the logarithm in $\omega_0(Y^{-1})$. This happens exactly for the value of n where the strings describing the Wilson loop can be replaced by D2-branes, which are the string theory incarnation of the vortex loop operators [18]. This suggests that the vortex loop operators are related to eigenvalues along the logarithmic branch-cut. It is possible to use our formalism to calculate the perturbative and instanton corrections to these configurations and it would be interesting to understand further their significance in the matrix model.

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A. Normalization of the ABJM Matrix Model

Here we shall fix the overall normalization of the matrix model. As explained in the beginning of Sect. 2, to do this we must fix the coefficient of the cosh in the denominator. This term appears as a consequence of integrating out the matter hypermultiplets at one-loop. For general supersymmetric Chern–Simons-matter theories, the contribution of a hypermultiplet in representation R is given by [4]

$$\log Z[a] = \log \prod_{\rho} \prod_{n=1}^{\infty} \left(\frac{n + 1/2 + i\rho(a)}{n - 1/2 - i\rho(a)} \right)^n, \quad (A.1)$$

where ρ are the weights of the representation, and a is the element in the Cartan algebra given by

$$a = \frac{1}{2\pi} \text{diag}(\mu_1, \dots, \mu_{N_1}, \nu_1, \dots, \nu_{N_2}). \quad (A.2)$$

In [4] the one-loop determinant is evaluated up to a multiplicative constant,

$$Z[a] = \prod_{\rho} (C \cosh(\pi \rho(a)))^{-1/2}. \quad (\text{A.3})$$

The constant C can be determined by setting $a = 0$ in (A.1)

$$-\frac{1}{2} \log C = \log \prod_{n=1}^{\infty} \left(\frac{n+1/2}{n-1/2} \right)^n. \quad (\text{A.4})$$

This is a divergent constant, but as usual when considering determinants on compact manifolds, we can compute it by using ζ -function regularization. Let us define

$$\zeta_Z(s) = \sum_{n=1}^{\infty} \left(\frac{n}{(n+\frac{1}{2})^s} - \frac{n}{(n-\frac{1}{2})^s} \right). \quad (\text{A.5})$$

The regularization of the quantity appearing in (A.4) is then $-\zeta'_Z(0)$. An elementary calculation shows that

$$\zeta_Z(s) = -(2^s - 1) \zeta(s), \quad (\text{A.6})$$

where $\zeta(s)$ is the standard Riemann zeta function. Therefore,

$$-\zeta'_Z(0) = -\frac{\log 2}{2} \quad (\text{A.7})$$

and $C = 2$.

B. Giant Wilson Loops in Chern–Simons Theory

Chern–Simons theory on \mathbb{S}^3 is a particular case of the lens space matrix model when $b = 1$ and the second cut collapses to zero size, *i.e.*, $t_1 = t$, $t_2 = 0$. It gives the leading behavior of the Wilson loop in ABJM theory when $\lambda_2 \ll \lambda_1$, as discussed in Sect. 6.

Here we consider the behavior of the giant Wilson loops, those in high dimensional symmetric or antisymmetric representations presented in Sect. 8.2, in this limit. In this case it is easy to calculate explicitly the action (8.35), since the integral

$$g(Y) = -\int_0^Y \frac{dY'}{Y'} \log(h(Y')), \quad h(Y) = \frac{1}{2} \left[1 + Y + \sqrt{(1+Y)^2 - 4e^t Y} \right] \quad (\text{B.1})$$

can be obtained in closed form

$$g(Y) = \frac{\pi^2}{6} - \frac{1}{2} \log^2(h(Y)) + \log(h(Y)) (\log(1 - e^{-t} h(Y)) - \log(1 - h(Y))) \\ - \text{Li}_2(h(Y)) + \text{Li}_2(e^{-t} h(Y)) - \text{Li}_2(e^{-t}). \quad (\text{B.2})$$

Here we used the dilogarithm identity

$$\text{Li}_2(1-x) = \frac{\pi^2}{6} - \text{Li}_2(x) - \log(x) \log(1-x). \quad (\text{B.3})$$

The solution of the saddle point Eqs. (8.25) is obtained by setting in (8.33)

$$\kappa = -4i \sinh \frac{t}{2}, \quad B = \frac{t}{2\pi i} + \frac{1}{2}, \quad (\text{B.4})$$

and we find

$$Y_* = -\frac{1 - e^{-2\pi i v}}{1 - e^{2\pi i v + t}}. \quad (\text{B.5})$$

The action (8.35) is

$$\begin{aligned} \eta A_\eta &= -2\pi i v \log(\eta Y_*) + g(Y_*) \\ &= -2\pi i v \log \eta - 2\pi^2 v^2 + 2\pi i v t + \frac{\pi^2}{6} \\ &\quad + \text{Li}_2(e^{2\pi i v - t}) - \text{Li}_2(e^{2\pi i v}) - \text{Li}_2(e^{-t}). \end{aligned} \quad (\text{B.6})$$

Notice that this expression is exact in t .

We can test (B.6) in all details against a direct calculation of correlators. Indeed, the VEVs $\langle \text{Tr}_R U \rangle$ for the Chern–Simons matrix model on \mathbb{S}^3 are proportional to quantum dimensions (see for example [21]):

$$\langle \text{Tr}_R U \rangle = q^{\kappa_R/2 + \ell(R)N/2} \dim_q(R). \quad (\text{B.7})$$

In this equation,

$$q = e^{g_s}, \quad (\text{B.8})$$

$\ell(R)$ is the number of boxes in R , and κ_R is the framing factor, given by

$$\kappa_R = \sum_i l_i(l_i - 2i + 1), \quad (\text{B.9})$$

where l_i are the lengths of the rows in the diagrams. The quantum dimensions of the symmetric and antisymmetric representations are given by

$$\dim_q(\mathcal{R}_n^\eta) = \frac{q^{\eta n(n-1)/4} e^{\eta t/2}}{[n]!} \prod_{i=1}^n (1 - e^{-t} q^{-\eta(i-1)}), \quad (\text{B.10})$$

where

$$[n]! = \prod_{i=1}^n (q^{i/2} - q^{-i/2}) = q^{\frac{1}{4}n(n+1)} \prod_{i=1}^n (1 - q^{-i}). \quad (\text{B.11})$$

At large n we rescale

$$\xi = \frac{i}{n}, \quad q^{-i} = \exp(-g_s i) \rightarrow e^{-2\pi i \eta v \xi} \quad (\text{B.12})$$

so that

$$\log([n]!) \approx \frac{1}{g_s} \left(-\pi^2 v^2 + 2\pi i \eta v \int_0^1 d\xi \log(1 - e^{-2\pi i \eta v \xi}) \right). \quad (\text{B.13})$$

This gives the following contribution to the action:

$$\begin{aligned} & \pi^2 v^2 + \frac{\pi^2}{6} - \text{Li}_2 \left(e^{-2\pi i \eta v} \right) \\ &= \eta \left(\pi^2 v^2 - 2\pi i v \log \eta + \frac{\pi^2}{6} - \text{Li}_2 \left(e^{-2\pi i v} \right) \right). \end{aligned} \quad (\text{B.14})$$

To derive the expression on the right hand side we used, for $\eta = -1$ the dilogarithm identity

$$\text{Li}_2(e^x) = -\text{Li}_2(e^{-x}) + \frac{\pi^2}{3} - \frac{x^2}{2} \pm \pi i x. \quad (\text{B.15})$$

The product in the numerator of both the symmetric and antisymmetric representations can be written in a unified form as

$$2\pi i \eta v \int_0^1 d\xi \log(1 - e^{-t} e^{-2\pi i v \xi}) = \eta \left(\text{Li}_2(e^{-t-2\pi i v}) - \text{Li}_2(e^{-t}) \right). \quad (\text{B.16})$$

The prefactors in (B.7) and (B.10) contribute

$$\eta(-3\pi^2 v^2 + 2\pi i v t). \quad (\text{B.17})$$

Together with (B.14) and (B.16) this exactly reproduces (B.6).

In the antisymmetric representation the result can also be written as

$$-2\pi i v(t + 2\pi i v) + \frac{\pi^2}{6} + \text{Li}_2(e^{-t}) - \text{Li}_2(e^{-t-2\pi i v}) - \text{Li}_2(e^{2\pi i v}). \quad (\text{B.18})$$

This expression agrees at leading order with the D6-brane calculation (8.21) and should be the full answer in the limit of $\lambda_2 = 0$. In this expression we see the expected symmetry [68]

$$n \leftrightarrow N - n, \quad (\text{B.19})$$

which is

$$2\pi i v \leftrightarrow -t - 2\pi i v. \quad (\text{B.20})$$

References

1. Aharony, O., Bergman, O., Jafferis, D.L., Maldacena, J.: $\mathcal{N} = 6$ superconformal Chern–Simons–matter theories, M2-branes and their gravity duals. *JHEP* **0810**, 091 (2008)
2. Gustavsson, A.: Algebraic structures on parallel M2-branes. *Nucl. Phys. B* **811**, 66 (2009)
3. Bagger, J., Lambert, N.: Gauge symmetry and supersymmetry of multiple M2-branes. *Phys. Rev. D* **77**, 065008 (2008)
4. Kapustin, A., Willett, B., Yaakov, I.: Exact results for Wilson loops in superconformal Chern–Simons theories with matter. *JHEP* **1003**, 089 (2010)
5. Pestun, V.: Localization of gauge theory on a four-sphere and supersymmetric Wilson loops. <https://arxiv.org/abs/0712.2824v2> [hep-th], 2010
6. Kapustin, A., Willett, B., Yaakov, I.: Nonperturbative Tests of Three-Dimensional Dualities. <https://arxiv.org/abs/1003.5694v2> [hep-th], 2010
7. Drukker, N., Trancanelli, D.: A supermatrix model for $\mathcal{N} = 6$ super Chern–Simons–matter theory. *JHEP* **1002**, 058 (2010)
8. Mariño, M., Putrov, P.: Exact results in ABJM theory from topological strings. *JHEP* **1006**, 011 (2010)

9. Mariño, M.: Chern–Simons theory, matrix integrals, and perturbative three-manifold invariants. *Commun. Math. Phys.* **253**, 25 (2004)
10. Aganagic, M., Klemm, A., Mariño, M., Vafa, C.: Matrix model as a mirror of Chern–Simons theory. *JHEP* **0402**, 010 (2004)
11. Aharony, O., Bergman, O., Jafferis, D.L.: Fractional M2-branes. *JHEP* **0811**, 043 (2008)
12. Witten, E.: Quantum field theory and the Jones polynomial. *Commun. Math. Phys.* **121**, 351 (1989)
13. Halmagyi, N., Yasnov, V.: The spectral curve of the lens space matrix model. *JHEP* **0911**, 104 (2009)
14. Mariño, M., Pasquetti, S., Putrov, P.: Large N duality beyond the genus expansion. <http://arXiv.org/abs/0911.4692v2> [hep-th], 2010
15. Drukker, N., Plefka, J., Young, D.: Wilson loops in 3-dimensional $\mathcal{N} = 6$ supersymmetric Chern–Simons Theory and their string theory duals. *JHEP* **0811**, 019 (2008)
16. Chen, B., Wu, J.B.: Supersymmetric Wilson loops in $\mathcal{N} = 6$ super Chern–Simons-matter theory. *Nucl. Phys. B* **825**, 38 (2010)
17. Rey, S.J., Suyama, T., Yamaguchi, S.: Wilson loops in superconformal Chern–Simons theory and fundamental strings in Anti-de Sitter supergravity dual. *JHEP* **0903**, 127 (2009)
18. Drukker, N., Gomis, J., Young, D.: Vortex Loop Operators, M2-branes and Holography. *JHEP* **0903**, 004 (2009)
19. Lee, K.M., Lee, S.: 1/2-BPS Wilson Loops and Vortices in ABJM Model. *JHEP* **1009**, 004 (2010)
20. Tierz, M.: Soft matrix models and Chern–Simons partition functions. *Mod. Phys. Lett. A* **19**, 1365 (2004)
21. Mariño, M.: Les Houches lectures on matrix models and topological strings. <http://arXiv.org/abs/hep-th/0410165v3>, 2010
22. Dolivet, Y., Tierz, M.: Chern–Simons matrix models and Stieltjes-Wigert polynomials. *J. Math. Phys.* **48**, 023507 (2007)
23. Bergman, O., Hirano, S.: Anomalous radius shift in $\text{AdS}_4/\text{CFT}_3$. *JHEP* **0907**, 016 (2009)
24. Aharony, O., Hashimoto, A., Hirano, S., Ouyang, P.: D-brane charges in gravitational duals of 2+1 dimensional gauge theories and duality cascades. *JHEP* **1001**, 072 (2010)
25. Klebanov, I.R., Tseytlin, A.A.: Entropy of near-extremal black p-branes. *Nucl. Phys. B* **475**, 164 (1996)
26. Minahan, J.A., Sax, O.O., Sieg, C.: A limit on the ABJ model. <http://arXiv.org/abs/1005.1786v1> [hep-th], 2010
27. Gadde, A., Pomoni, E., Rastelli, L.: Spin chains in $\mathcal{N} = 2$ superconformal theories: from the \mathbb{Z}_2 quiver to superconformal QCD. <http://arXiv.org/abs/1006.0015v1> [hep-th], 2010
28. Huang, M.x., Klemm, A.: Holomorphic anomaly in gauge theories and matrix models. *JHEP* **0709**, 054 (2007)
29. Grimm, T.W., Klemm, A., Mariño, M., Weiss, M.: Direct integration of the topological string. *JHEP* **0708**, 058 (2007)
30. Bershadsky, M., Cecotti, S., Ooguri, H., Vafa, C.: Kodaira–Spencer theory of gravity and exact results for quantum string amplitudes. *Commun. Math. Phys.* **165**, 311 (1994)
31. Haghighat, B., Klemm, A., Rauch, M.: Integrability of the holomorphic anomaly equations. *JHEP* **0810**, 097 (2008)
32. Shenker, S.H.: The Strength of nonperturbative effects in string theory. In: Álvarez, O., Marinari, E., Windey, P. (eds.) *Random surfaces and quantum gravity*. New York: Plenum 1991, pp. 191–200
33. Drukker, N., Fiol, B.: All-genus calculation of Wilson loops using D-branes. *JHEP* **0502**, 010 (2005)
34. Gomis, J., Passerini, F.: Holographic Wilson loops. *JHEP* **0608**, 074 (2006)
35. Gomis, J., Passerini, F.: Wilson loops as D3-branes. *JHEP* **0701**, 097 (2007)
36. Kapustin, A., Saulina, N.: Chern–Simons–Rozansky–Witten topological field theory. *Nucl. Phys. B* **823**, 403 (2009)
37. Brini, A., Tanzini, A.: Exact results for topological strings on resolved $Y(p, q)$ singularities. *Commun. Math. Phys.* **289**, 205 (2009)
38. Erickson, J.K., Semenoff, G.W., Zarembo, K.: Wilson loops in $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. *Nucl. Phys. B* **582**, 155 (2000)
39. Hosono, S., Klemm, A., Theisen, S.: Lectures on mirror symmetry. In: *Lecture Notes in Phys.*, Vol. **436**, Berlin: Heidelberg–New York: Springer, 1994, pp. 235–280
40. Chiang, T.M., Klemm, A., Yau, S.T., Zaslow, E.: Local mirror symmetry: Calculations and interpretations. *Adv. Theor. Math. Phys.* **3**, 495 (1999)
41. Aganagic, M., Bouchard, V., Klemm, A.: Topological strings and (almost) modular forms. *Commun. Math. Phys.* **277**, 771 (2008)
42. Seiberg, N., Witten, E.: Electric–magnetic duality, monopole condensation, and confinement in $\mathcal{N} = 2$ supersymmetric Yang–Mills theory. *Nucl. Phys. B* **426**, 19 (1994) [Erratum-ibid. *B* **430**, 485 (1994)]
43. Guadagnini, E., Martellini, M., Mintchev, M.: Wilson lines in Chern–Simons theory and link invariants. *Nucl. Phys. B* **330**, 575 (1990)
44. Alvarez, M., Labastida, J.M.F.: Analysis of observables in Chern–Simons perturbation theory. *Nucl. Phys. B* **395**, 198 (1993)

45. Drukker, N., Gross, D.J.: An exact prediction of $\mathcal{N} = 4$ SUSYM theory for string theory. *J. Math. Phys.* **42**, 2896 (2001)
46. Drukker, N., Gross, D.J., Tseytlin, A.A.: Green-Schwarz string in $\text{AdS}_5 \times \mathbb{S}^5$: Semiclassical partition function. *JHEP* **0004**, 021 (2000)
47. Kruczenski, M., Tirziu, A.: Matching the circular Wilson loop with dual open string solution at 1-loop in strong coupling. *JHEP* **0805**, 064 (2008)
48. Drukker, N.: 1/4 BPS circular loops, unstable world-sheet instantons and the matrix model. *JHEP* **0609**, 004 (2006)
49. Zarembo, K.: Supersymmetric Wilson loops. *Nucl. Phys. B* **643**, 157 (2002)
50. Bak, D., Yun, S.: Thermal aspects of ABJM theory: Currents and condensations. *Class. Quant. Grav.* **27**, 215011 (2010)
51. Balasubramanian, V., Kraus, P.: A stress tensor for anti-de Sitter gravity. *Commun. Math. Phys.* **208**, 413 (1999)
52. Emparan, R., Johnson, C.V., Myers, R.C.: Surface terms as counterterms in the AdS/CFT correspondence. *Phys. Rev. D* **60**, 104001 (1999)
53. Cagnazzo, A., Sorokin, D., Wulff, L.: String instanton in $\text{AdS}_4 \times \mathbb{CP}^3$. *JHEP* **1005**, 009 (2010)
54. Giombi, S., Pestun, V., Ricci, R.: Notes on supersymmetric Wilson loops on a two-sphere. <http://arXiv.org/abs/0905.0665v2> [hep-th], 2009
55. Arsiwalla, X., Boels, R., Mariño, M., Sinkovics, A.: Phase transitions in q-deformed 2d Yang-Mills theory and topological strings. *Phys. Rev. D* **73**, 026005 (2006)
56. Ambjorn, J., Chekhov, L., Kristjansen, C.F., Makeenko, Yu.: Matrix model calculations beyond the spherical limit. *Nucl. Phys. B* **404**, 127 (1993) [Erratum-ibid. *B* **449**, 681 (1995)]
57. Alday, L.F., Eden, B., Korchemsky, G.P., Maldacena, J., Sokatchev, E.: From correlation functions to Wilson loops. <http://arXiv.org/abs/1007.3243v2> [hep-th], 2010
58. Klemm, A., Mariño, M., Rauch, M.: Direct integration and non-perturbative effects in matrix models. *JHEP* **1010**, 004 (2010)
59. Eynard, B., Orantin, N.: Invariants of algebraic curves and topological expansion. <http://arXiv.org/abs/math-ph/0702045v4>, 2007
60. Mariño, M.: Open string amplitudes and large order behavior in topological string theory. *JHEP* **0803**, 060 (2008)
61. Bouchard, V., Klemm, A., Mariño, M., Pasquetti, S.: Remodeling the B-model. *Commun. Math. Phys.* **287**, 117 (2009)
62. Huang, M.x., Klemm, A.: Holomorphicity and modularity in Seiberg-Witten theories with matter. <http://arXiv.org/abs/0902.1325v1> [hep-th], 2009
63. Eynard, B., Mariño, M., Orantin, N.: Holomorphic anomaly and matrix models. *JHEP* **0706**, 058 (2007)
64. Mariño, M., Schiappa, R., Weiss, M.: Nonperturbative effects and the large-order behavior of matrix models and topological strings. *Commun. Number Theor. Phys.* **2**, 349 (2008)
65. Pasquetti, S., Schiappa, R.: Borel and Stokes nonperturbative phenomena in topological string theory and $c = 1$ matrix models. <http://arXiv.org/abs/0907.4082v2> [hep-th], 2010
66. Akemann, G.: Higher genus correlators for the Hermitian matrix model with multiple cuts. *Nucl. Phys. B* **482**, 403 (1996)
67. Klemm, A., Mariño, M., Theisen, S.: Gravitational corrections in supersymmetric gauge theory and matrix models. *JHEP* **0303**, 051 (2003)
68. Yamaguchi, S.: Wilson loops of anti-symmetric representation and D5-branes. *JHEP* **0605**, 037 (2006)
69. Lunin, O.: 1/2-BPS states in M theory and defects in the dual CFTs. *JHEP* **0710**, 014 (2007)
70. Hartnoll, S.A., Kumar, S.P.: Higher rank Wilson loops from a matrix model. *JHEP* **0608**, 026 (2006)
71. Lin, H., Lunin, O., Maldacena, J.M.: Bubbling AdS space and 1/2 BPS geometries. *JHEP* **0410**, 025 (2004)
72. Bars, I.: *Supergroups and their representations*. In: Application of Group Theory in Phys. and Math. Phys., Lectures Appl. Math. **21**, Providence, RI: Amer. Math. Soc., 1985, p. 17